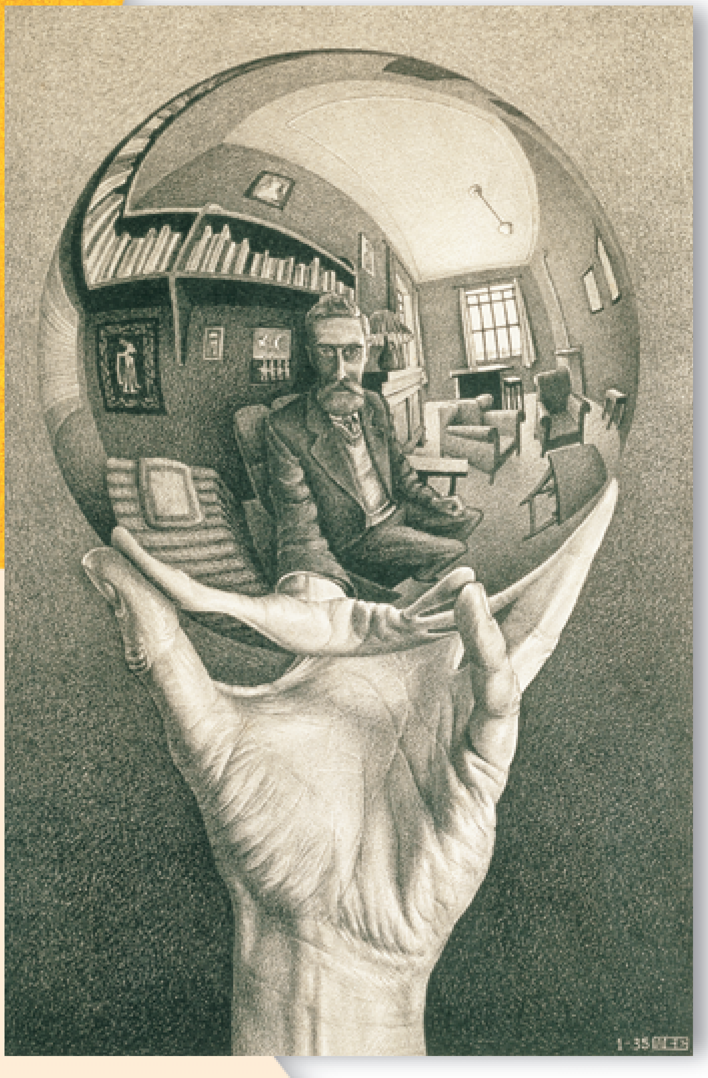
CHAPTER

2 Reasoning in Geometry



That which an artist makes is a mirror image of what he sees around him.

M. C. ESCHER

Hand with Reflecting Sphere (Self-Portrait in Spherical Mirror), M. C. Escher
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OBJECTIVES

In this chapter you will

- perform geometry investigations and make many discoveries by observing common features or patterns
- use your discoveries to solve problems through a process called inductive reasoning
- use inductive reasoning to discover patterns
- learn to use deductive reasoning
- learn about vertical angles and linear pairs
- make conjectures

LESSON

2.1

Inductive Reasoning

We have to reinvent the wheel every once in a while, not because we need a lot of wheels; but because we need a lot of inventors.

BRUCE JOYCE

As a child you learned by experimenting with the natural world around you. You learned how to walk, to talk, and to ride your first bicycle, all by trial and error. From experience you learned to turn a water faucet on with a counterclockwise motion and to turn it off with a clockwise motion. You achieved most of your learning by a process called **inductive reasoning.** It is the process of observing data, recognizing patterns, and making generalizations about those patterns.

Geometry is rooted in inductive reasoning. In ancient Egypt and Babylonia, geometry began when people developed procedures for measurement after much experience and observation. Assessors and surveyors used these procedures to calculate land areas and to reestablish the boundaries of agricultural fields after floods. Engineers used the procedures to build canals, reservoirs, and the Great Pyramids. Throughout this course you will use inductive reasoning. You will perform investigations, observe similarities and patterns, and make many discoveries that you can use to solve problems.

Language CONNECTION

The word "geometry" means "measure of the earth" and was originally inspired by the ancient Egyptians. The ancient Egyptians devised a complex system of land surveying in order to reestablish land boundaries that were erased each spring by the annual flooding of the Nile River.

Inductive reasoning guides scientists, investors, and business managers. All of these professionals use past experience to assess what is likely to happen in the future.

When you use inductive reasoning to make a generalization, the generalization is called a **conjecture.** Consider the following example from science.

EXAMPLE A

A scientist dips a platinum wire into a solution containing salt (sodium chloride), passes the wire over a flame, and observes that it produces an orange-yellow flame.

She does this with many other solutions that contain salt, finding that they all produce an orange-yellow flame. Make a conjecture based on her findings.

Platinum wire flame test

▶ Solution

The scientist tested many other solutions containing salt, and found no counterexamples. You should conjecture: "If a solution contains sodium chloride, then in a flame test it produces an orange-yellow flame."

Like scientists, mathematicians often use inductive reasoning to make discoveries. For example, a mathematician might use inductive reasoning to find patterns in a number sequence. Once he knows the pattern, he can find the next term.

EXAMPLE B

Consider the sequence

2, 4, 7, 11, . . .

Make a conjecture about the rule for generating the sequence. Then find the next three terms.

▶ Solution

Look at the numbers you add to get each term. The 1st term in the sequence is 2. You add 2 to find the 2nd term. Then you add 3 to find the 3rd term, and so on.

$$2, \xrightarrow{+2} 4, \xrightarrow{+3} 7, \xrightarrow{+4} 11$$

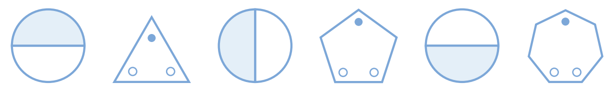
You can conjecture that if the pattern continues, you always add the next counting number to get the next term. The next three terms in the sequence will be 16, 22, and 29.

$$11, \xrightarrow{+5} 16, \xrightarrow{+6} 22, \xrightarrow{+7} 29$$

In the following investigation you will use inductive reasoning to recognize a pattern in a series of drawings and use it to find a term much farther out in a sequence.

Investigation
Shape Shifters

Look at the sequence of shapes below. Pay close attention to the patterns that occur in every other shape.



Step 1 What patterns do you notice in the 1st, 3rd, and 5th shapes?

Step 2 What patterns do you notice in the 2nd, 4th, and 6th shapes?

Step 3 Draw the next two shapes in the sequence.

Step 4 Use the patterns you discovered to draw the 25th shape.

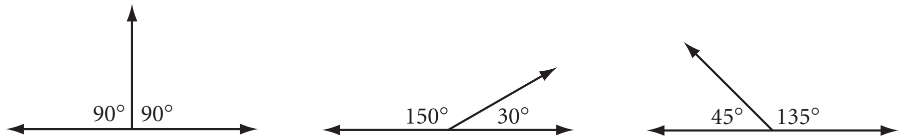
Step 5 Describe the 30th shape in the sequence. You do not have to draw it!

Sometimes a conjecture is difficult to find because the data collected are unorganized or the observer is mistaking coincidence with cause and effect. Good use of inductive reasoning depends on the quantity and quality of data. Sometimes not enough information or data have been collected to make a proper conjecture. For example, if you are asked to find the next term in the pattern 3, 5, 7, you might conjecture that the next term is 9—the next odd number. Someone else might notice that the pattern is the consecutive odd primes and say that the next term is 11. If the pattern was 3, 5, 7, 11, 13, what would you be more likely to conjecture?

EXERCISES

1. On his way to the local Hunting and Gathering Convention, caveperson Stony Grok picks up a rock, drops it into a lake, and notices that it sinks. He picks up a second rock, drops it into the lake, and notices that it also sinks. He does this five more times. Each time, the rock sinks straight to the bottom of the lake. Stony conjectures: "Ura nok seblu," which translates to _?_. What counterexample would Stony Grok need to find to disprove, or at least to refine, his conjecture? ⓗ

2. Sean draws these geometric figures on paper. His sister Courtney measures each angle with a protractor. They add the measures of each pair of angles to form a conjecture. Write their conjecture.



For Exercises 3–10, use inductive reasoning to find the next two terms in each sequence.

3. 1, 10, 100, 1000, ?, ?

4. $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, ?, ? ⓗ

5. 7, 3, -1, -5, -9, -13, ?, ?

6. 1, 3, 6, 10, 15, 21, ?, ?

7. 1, 1, 2, 3, 5, 8, 13, ?, ? ⓗ

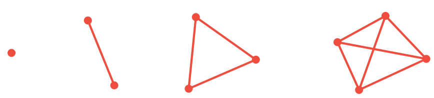
8. 1, 4, 9, 16, 25, 36, ?, ? ⓗ

9. 32, 30, 26, 20, 12, 2, ?, ?

10. 1, 2, 4, 8, 16, 32, ?, ?

For Exercises 11–16, use inductive reasoning to draw the next shape in each picture pattern.

11.



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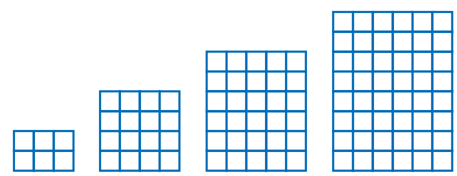
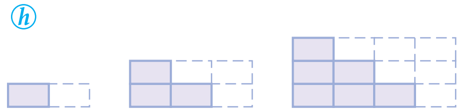
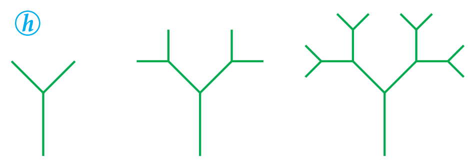


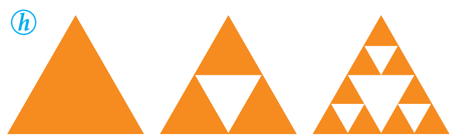
13. ⓗ



14. ⓗ



15. ⓗ



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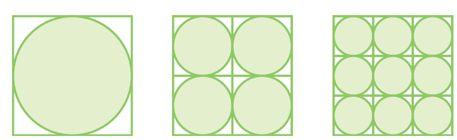


Use the rule provided to generate the first five terms of the sequence in Exercise 17 and the next five terms of the sequence in Exercise 18.

17. $3n - 2$ ⓗ

18. $1, 3, 6, 10, \ldots, \frac{n(n+1)}{2}, \ldots$

19. Now it's your turn. Generate the first five terms of a sequence. Give the sequence to a member of your family or to a friend and ask him or her to find the next two terms in the sequence. Can he or she find your pattern?

20. Write the first five terms of two different sequences in which 12 is the 3rd term.

21. Think of a situation in which you have used inductive reasoning. Write a paragraph describing what happened and explaining why you think it was inductive reasoning. ⓗ

22. Look at the pattern in these pairs of equations. Decide if the conjecture is true. If it is not true, find a counterexample.

$12^2 = 144$ and $21^2 = 441$

$13^2 = 169$ and $31^2 = 961$

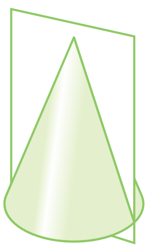
$103^2 = 10609$ and $301^2 = 90601$

$112^2 = 12544$ and $211^2 = 44521$

Conjecture: If two numbers have the same digits in reverse order, then the squares of those numbers will have identical digits but in reverse order.

Review

23. Sketch the section formed when the cone is sliced by the plane, as shown.



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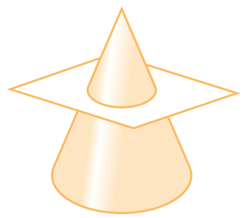
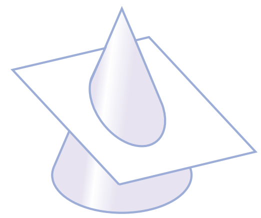
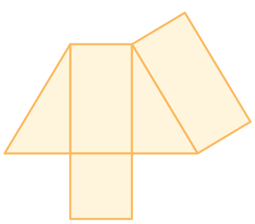


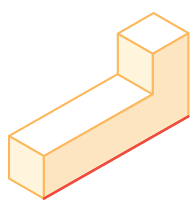
25.



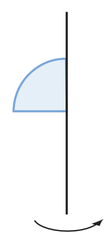
26. Sketch the three-dimensional figure formed by folding the net below into a solid.



27. Sketch the figure shown below but with the red edge vertical and facing you.



28. Sketch the solid of revolution formed when the two-dimensional figure is rotated about the line.



For Exercises 29–38, write the word that makes the statement true.

29. Points are _?_ if they lie on the same line.

30. A triangle with two congruent sides is _?_.

31. The geometry tool used to measure the size of an angle in degrees is called a(n) _?_.

32. A(n) _?_ of a circle connects its center to a point on the circle.

33. A segment connecting any two non-adjacent vertices in a polygon is called a(n) _?_.

34. A polygon with 12 sides is called a(n) _?_.

35. A trapezoid has exactly one pair of _?_ sides.

36. A(n) $\underline{\ ?\ }$ polygon is both equiangular and equilateral.

37. If angles are complementary, then their measures add to $\underline{\ ?\ }$.

38. If two lines intersect to form a right angle, then they are $\underline{\ ?\ }$.

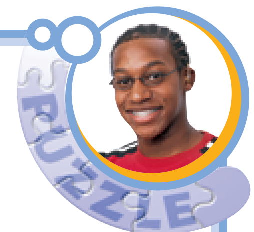
For Exercises 39–42, sketch and label the figure.

39. Pentagon *GIANT* with diagonal $\overline{AG}$ parallel to side $\overline{NT}$

40. A quadrilateral that has reflectional symmetry but not rotational symmetry

41. A prism with a hexagonal base

42. A counterexample to show that the following statement is false: The diagonals of a kite bisect the angles. ⓗ



IMPROVING YOUR REASONING SKILLS

Puzzling Patterns

These patterns are "different." Your task is to find the next term.

1. 18, 46, 94, 63, 52, 61, $\underline{\ ?\ }$

2. O, T, T, F, F, S, S, E, N, $\underline{\ ?\ }$

3. 1, 4, 3, 16, 5, 36, 7, $\underline{\ ?\ }$

4. 4, 8, 61, 221, 244, 884, $\underline{\ ?\ }$

5. 6, 8, 5, 10, 3, 14, 1, $\underline{\ ?\ }$

6. **B**, **0**, **C**, **2**, **D**, **0**, **E**, **3**, **F**, **3**, **G**, $\underline{\ ?\ }$

7. 2, 3, 6, 1, 8, 6, 8, 4, 8, 4, 8, 3, 2, 3, 2, 3, $\underline{\ ?\ }$

8. A E F H I K L M N T V W

B C D G J O P Q R S U

Where do the X, Y, and Z go?

LESSON

2.2

Deductive Reasoning

That's the way things come clear. All of a sudden. And then you realize how obvious they've been all along.

MADELEINE L'ENGLE

Have you ever noticed that the days are longer in the summer? Or that mosquitoes appear after a summer rain? Over the years you have made conjectures, using inductive reasoning, based on patterns you have observed. When you make a conjecture, the process of discovery may not always help explain *why* the conjecture works. You need another kind of reasoning to help answer this question.

Deductive reasoning is the process of showing that certain statements follow logically from agreed-upon assumptions and proven facts. When you use deductive reasoning, you try to reason in an orderly way to convince yourself or someone else that your conclusion is valid. If your initial statements are true, and you give a logical argument, then you have shown that your conclusion is true. For example, in a trial, lawyers use deductive arguments to show how the evidence that they present proves their case. A lawyer might make a very good argument. But first, the court must believe the evidence and accept it as true.

The success of an attorney's case depends on the jury accepting the evidence as true and following the steps in her deductive reasoning.

You use deductive reasoning in algebra. When you provide a reason for each step in the process of solving an equation, you are using deductive reasoning. Here is an example.

EXAMPLE A

Solve the equation for x. Give a reason for each step in the process.

$$3(2x + 1) + 2(2x + 1) + 7 = 42 - 5x$$

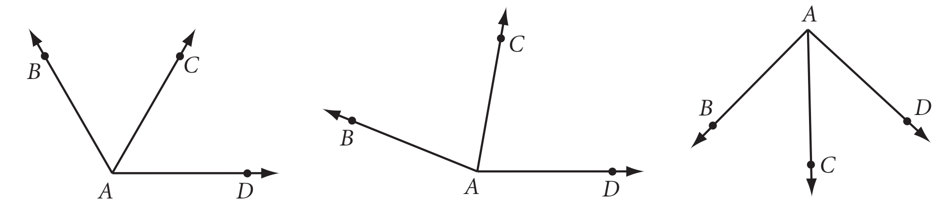
▶ Solution

$3(2x + 1) + 2(2x + 1) + 7 = 42 - 5x$	The original equation.
$5(2x + 1) + 7 = 42 - 5x$	Combining like terms.
$5(2x + 1) = 35 - 5x$	Subtraction property of equality.
$10x + 5 = 35 - 5x$	Distributive property.
$10x = 30 - 5x$	Subtraction property of equality.
$15x = 30$	Addition property of equality.
$x = 2$	Division property of equality.

The next example shows how to use both kinds of reasoning: inductive reasoning to discover the property and deductive reasoning to explain why it works.

EXAMPLE B

In each diagram, $\overrightarrow{AC}$ bisects obtuse angle BAD. Classify $\angle BAD$, $\angle DAC$, and $\angle CAB$ as *acute, right,* or *obtuse*. Then complete the conjecture.



Conjecture: If an obtuse angle is bisected, then the two newly formed congruent angles are ___?___.

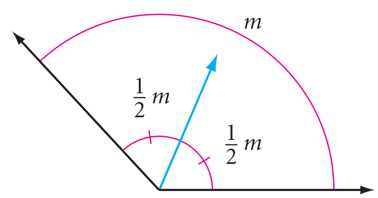
Justify your answers with a deductive argument.

▶ Solution

In each diagram, $\angle BAD$ is obtuse because $m\angle BAD$ is greater than 90°. In each diagram, the angles formed by the bisector are acute because their measures—60°, 79°, and 46°—are less than 90°. So one possible conjecture is

Conjecture: If an obtuse angle is bisected, then the two newly formed congruent angles are *acute*.

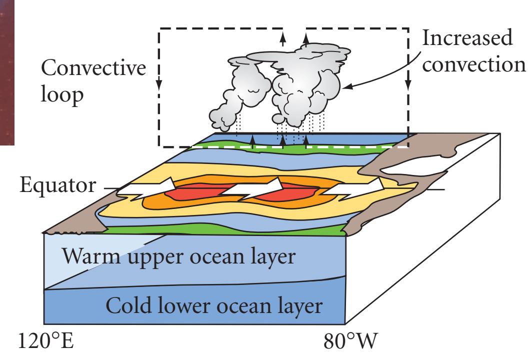
Why? According to our definition of an angle, every angle measure is less than 180°. So, using algebra, if m is the measure of an obtuse angle, then $m < 180°$. When you bisect an angle, the two newly formed angles each measure half of the original angle, or $\frac{1}{2}m$. If $m < 180°$, then $\frac{1}{2}m < \frac{1}{2}(180)$, so $\frac{1}{2}m < 90°$. The two angles are each less than 90°, so they are acute.



Science CONNECTION

Here is an example of inductive reasoning, supported by deductive reasoning. El Niño is the warming of water in the tropical Pacific Ocean, which produces unusual weather conditions and storms worldwide. For centuries, farmers living in the Andes Mountains of South America have observed the stars in the Pleiades constellation to predict El Niño conditions. If the Pleiades look dim in June, they predict an El Niño year. What is the connection? Scientists have recently found that in an El Niño year, increased evaporation from the ocean produces high-altitude clouds that are invisible to the eye, but create a haze that makes stars more difficult to see. Therefore, the pattern that the Andean farmers knew about for centuries is now supported by a scientific explanation. To find out more about this story, go to www.keymath.com/DG.

El Niño Conditions

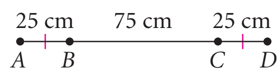


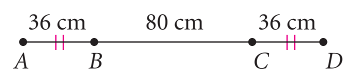
Inductive reasoning allows you to discover new ideas based on observed patterns. Deductive reasoning can help explain why your conjectures are true.

Inductive and deductive reasoning work very well together. In this investigation you will use inductive reasoning to form a conjecture and deductive reasoning to explain why it's true.

Investigation
Overlapping Segments

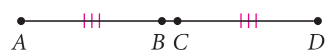
In each segment, $\overline{AB} \cong \overline{CD}$.





Step 1 From the markings on each diagram, determine the lengths of $\overline{AC}$ and $\overline{BD}$. What do you discover about these segments?

Step 2 Draw a new segment. Label it $\overline{AD}$. Place your own points B and C on $\overline{AD}$ so that $\overline{AB} \cong \overline{CD}$.



Step 3 Measure $\overline{AC}$ and $\overline{BD}$. How do these lengths compare?

Step 4 Complete the conclusion of this conjecture:

If $\overline{AD}$ has points A, B, C, and D in that order with $\overline{AB} \cong \overline{CD}$, then $\underline{\ ?\ }$.

Now you will use deductive reasoning and algebra to explain why the conjecture from Step 4 is true.

Step 5 Use deductive reasoning to convince your group that AC will always equal BD. Take turns explaining to each other. Write your argument algebraically.

In the investigation you used both inductive and deductive reasoning to convince yourself of the overlapping segments property. You will use a similar process in the next lesson to discover and prove the overlapping angles property in Exercise 17.

Good use of deductive reasoning depends on the quality of the argument. Just like the saying, "A chain is only as strong as its weakest link," a deductive argument is only as good (or as true) as the statements used in the argument. A conclusion in a deductive argument is true only if *all* the statements in the argument are true. Also, the statements in your argument must clearly follow from each other. Did you use clear arguments in explaining the investigation steps? Did you point out that $\overline{BC}$ is part of both $\overline{AC}$ and $\overline{BD}$? Did you point out that if you add the same amount to things that are equal the resulting sum must be equal?

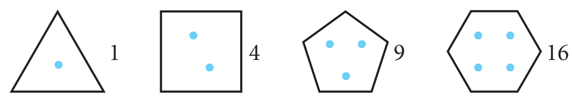
EXERCISES

1. When you use $\underline{\ ?\ }$ reasoning you are generalizing from careful observation that something is probably true. When you use $\underline{\ ?\ }$ reasoning you are establishing that, if a set of properties is accepted as true, something else must be true.

2. $\angle A$ and $\angle B$ are complementary. $m\angle A = 25°$. What is $m\angle B$? What type of reasoning do you use, inductive or deductive reasoning, when solving this problem?

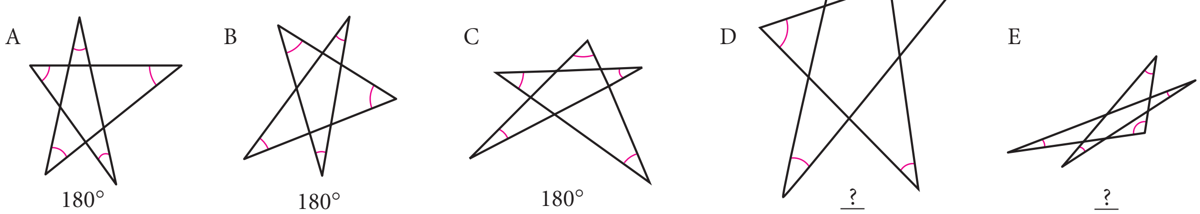
3. If the pattern continues, what are the next two terms?

What type of reasoning do you use, inductive or deductive reasoning, when solving this problem?



4. $\triangle DGT$ is isosceles with $TD = DG$. If the perimeter of $\triangle DGT$ is 756 cm and $GT = 240$ cm, then $DG = \underline{\ ?\ }$. What type of reasoning do you use, inductive or deductive reasoning, when solving this problem?

5. ***Mini-Investigation*** The sum of the measures of the five marked angles in stars A through C is shown below each star. Use your protractor to carefully measure the five marked angles in star D.



If this pattern continues, without measuring, what would be the sum of the measures of the marked angles in star E? What type of reasoning do you use, inductive or deductive reasoning, when solving this problem?

6. The definition of a parallelogram says, "If both pairs of opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram." Quadrilateral *LNDA* has both pairs of opposite sides parallel. What conclusion can you make? What type of reasoning did you use?

7. Use the overlapping segments property to complete each statement.

a. If $AB = 3$, then $CD = \underline{\ ?\ }$.
b. If $AC = 10$, then $BD = \underline{\ ?\ }$.
c. If $BC = 4$ and $CD = 3$, then $AC = \underline{\ ?\ }$.

8. In Example B of this lesson you discovered through inductive reasoning that if an obtuse angle is bisected, then the two newly formed congruent angles are acute. You then used deductive reasoning to explain why they were acute. Go back to the example and look at the sizes of the acute angles formed. What is the smallest possible size for the two congruent acute angles formed by the bisector? Can you use deductive reasoning to explain why? ⓗ

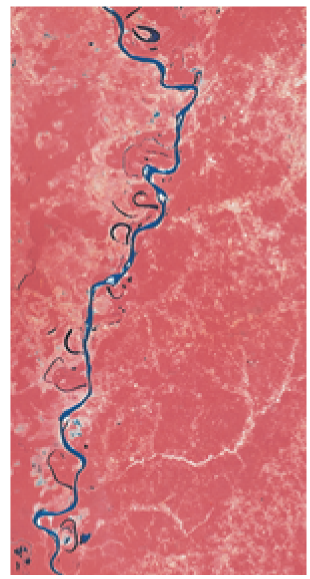
9. Study the pattern and make a conjecture by completing the fifth line. What would be the conjecture for the sixth line? The tenth line? (h)

$$\begin{aligned} 1 \cdot 1 &= 1 \\ 11 \cdot 11 &= 121 \\ 111 \cdot 111 &= 12{,}321 \\ 1{,}111 \cdot 1{,}111 &= 1{,}234{,}321 \\ 11{,}111 \cdot 11{,}111 &= \underline{\ ?\ } \end{aligned}$$

10. Think of a situation you observed outside of school in which deductive reasoning was used correctly. Write a paragraph or two describing what happened and explaining why you think it was deductive reasoning.

Review

11. Mark Twain once observed that the lower Mississippi River is very crooked and that over the years, as the bends and the turns straighten out, the river gets shorter and shorter. Using numerical data about the length of the lower part of the river, he noticed that in the year 1700 the river was more than 1200 miles long, yet by the year 1875 it was only 973 miles long. Twain concluded that any person "can see that 742 years from now the lower Mississippi will be only a mile and three-quarters long." What is wrong with this inductive reasoning?



Aerial photo of the Mississippi River

For Exercises 12–14, use inductive reasoning to find the next two terms of the sequence.

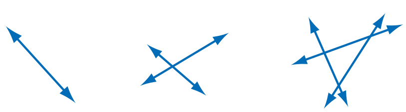
12. 180, 360, 540, 720, $\underline{\ ?\ }$, $\underline{\ ?\ }$ (h)

13. 0, 10, 21, 33, 46, 60, $\underline{\ ?\ }$, $\underline{\ ?\ }$

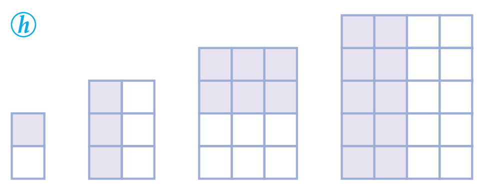
14. $\frac{1}{2}$, 9, $\frac{2}{3}$, 10, $\frac{3}{4}$, 11, $\underline{\ ?\ }$, $\underline{\ ?\ }$

For Exercises 15–18, draw the next shape in each picture pattern.

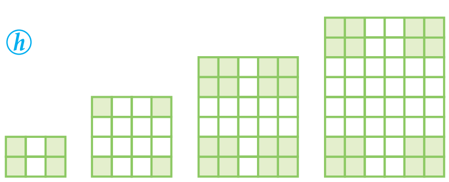
15.



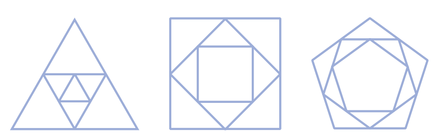
16. (h)



17. (h)



18.



19. Think of a situation you have observed in which inductive reasoning was used incorrectly. Write a paragraph or two describing what happened and explaining why you think it was an incorrect use of inductive reasoning.

Match each term in Exercises 20–29 with one of the figures A–O.

20. Kite

21. Consecutive angles in a polygon

22. Trapezoid

23. Diagonal in a polygon

24. Pair of complementary angles

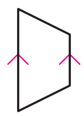
25. Radius

26. Pair of vertical angles

27. Chord

28. Acute angle

29. Angle bisector in a triangle

A.


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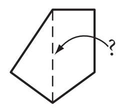

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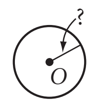

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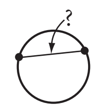

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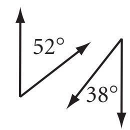

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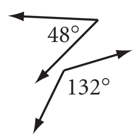

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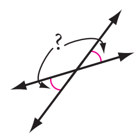

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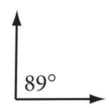

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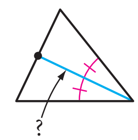

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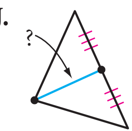

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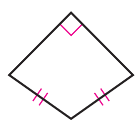

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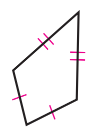

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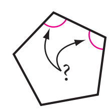

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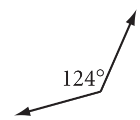

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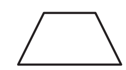


For Exercises 30–33, sketch and carefully label the figure.

30. Pentagon *WILDE* with $\angle ILD \cong \angle LDE$ and $\overline{LD} \cong \overline{DE}$

31. Isosceles obtuse triangle *OBG* with $m\angle BGO = 140°$

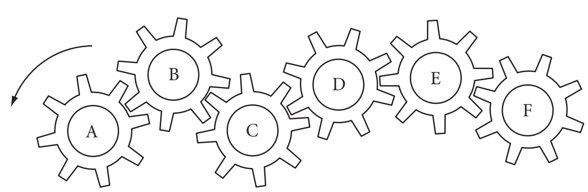
32. Circle *O* with a chord $\overline{CD}$ perpendicular to radius $\overline{OT}$

33. Circle *K* with angle *DIN* where *D*, *I*, and *N* are points on circle *K*

IMPROVING YOUR VISUAL THINKING SKILLS

Rotating Gears

In what direction will gear E rotate if gear A rotates in a counterclockwise direction?



LESSON

2.3

Finding the *n*th Term

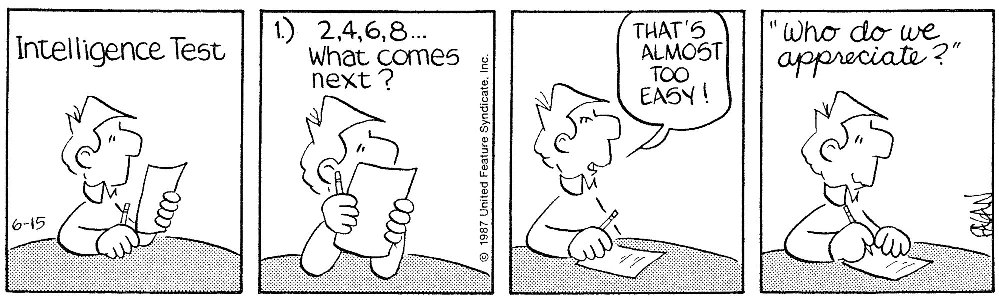
If you do something once, people call it an accident. If you do it twice, they call it a coincidence. But do it a third time and you've just proven a natural law.

GRACE MURRAY HOPPER

What would you do to get the next term in the sequence 20, 27, 34, 41, 48, 55, . . . ? A good strategy would be to find a pattern, using inductive reasoning. Then you would look at the differences between consecutive terms and predict what comes next. In this case there is a constant difference of +7. That is, you add 7 each time.

The next term is 55 + 7, or 62. What if you needed to know the value of the 200th term of the sequence? You certainly don't want to generate the next 193 terms just to get one answer. If you knew a rule for calculating *any* term in a sequence, without having to know the previous term, you could apply it to directly calculate the 200th term. The rule that gives the *n*th term for a sequence is called the **function rule.**

Let's see how the constant difference can help you find the function rule for some sequences.



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Investigation
Finding the Rule

Step 1 Copy and complete each table. Find the differences between consecutive values.

a.

n	1	2	3	4	5	6	7	8
$n - 5$	-4	-3	-2					

b.

n	1	2	3	4	5	6	7	8
$4n - 3$	1	5	9					

c.

n	1	2	3	4	5	6	7	8
$-2n + 5$	3	1	-1					

d.

n	1	2	3	4	5	6	7	8
$3n - 2$	1	4	7					

e.

n	1	2	3	4	5	6	7	8
$-5n + 7$	2	-3	-8					

Step 2 Did you spot the pattern? If a sequence has a constant difference of 4, then the number in front of the n (the coefficient of n) is $\underline{?}$. In general, if the difference between the values of consecutive terms of a sequence is always the same, say m (a constant), then the coefficient of n in the formula is $\underline{?}$.

Let's return to the sequence at the beginning of the lesson.

Term	1	2	3	4	5	6	7	...	n
Value	20	27	34	41	48	55	62	...	

+7 +7

The constant difference is 7, so you know part of the rule is $7n$. How do you find the rest of the rule?

Step 3 The first term ($n = 1$) of the sequence is 20, but if you apply the part of the rule you have so far using $n = 1$, you get $7n = 7(1) = 7$, not 20. So how should you fix the rule? How can you get from 7 to 20? What is the rule for this sequence?

Step 4 Check your rule by trying the rule with other terms in the sequence.

Let's look at an example of how to find a function rule, for the nth term in a number pattern.

EXAMPLE A Find the rule for the sequence 7, 2, −3, −8, −13, −18, . . .

▶ Solution

Placing the terms and values in a table we get

Term	1	2	3	4	5	6	...	n
Value	7	2	−3	−8	−13	−18	...	

The difference between the terms is always −5. So the rule is

$-5n +$ "something"

Let's use c to stand for the unknown "something." So the rule is

$-5n + c$

To find c, replace the n in the rule with a term number. Try $n = 1$ and set the expression equal to 7.

$$-5(1) + c = 7$$
$$c = 12$$

The rule is $-5n + 12$.

You can find the value of any term in the sequence by substituting the term number for n in the function rule. Let's look at an example of how to find the 200th term in a geometric pattern.

EXAMPLE B

If you place 200 points on a line, into how many non-overlapping rays and segments does it divide the line?

▶ Solution

Wait! don't start placing 200 points on a line. You need to find a rule that relates the number of points placed on a line to the number of parts created by those points. Then you can use your rule to answer the problem.

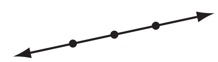
Start by creating a table.

Points dividing the line	1	2	3	4	5	6	...	n	...	200
Non-overlapping rays							...		...	
Non-overlapping segments							...		...	
Total							...		...	

Sketch one point dividing a line. One point gives you just two rays. Enter that into the table.

Next, sketch two points dividing a line. This gives one segment and the two end rays. Enter the value into your table.

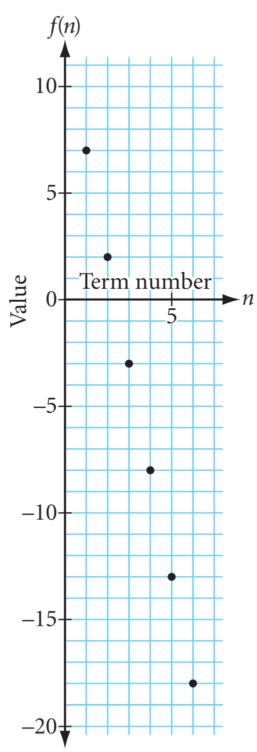
Next, sketch three points dividing a line, then four, then five, and so on. The table completed for one to three points is



Points dividing the line	1	2	3	4	5	6	...	n	...	200
Non-overlapping rays	2	2	2				...		...	
Non-overlapping segments	0	1	2				...		...	
Total	2	3	4				...		...	

Once you have found values for 1, 2, 3, 4, 5, and 6 points on a line you next try to find the rule for each sequence. There are always two non-overlapping rays so for 200 points there will be two rays. The rule, or nth term, for the number of non-overlapping segments is $n - 1$. For 200 points there will be 199 segments. The rule, or nth term, for the total number of distinct rays and segments of the line is $n + 1$. For 200 points there will be 201 distinct parts of the line.

This process of looking at patterns and generalizing a rule, or nth term, is inductive reasoning. To understand why the rule is what it is, you can turn to deductive reasoning. Notice that adding another point on a line divides a segment into two segments. So each new point adds one more segment to the pattern.



Rules that generate a sequence with a constant difference are **linear functions.** To see why they're called linear, you can graph the term number and the value for the sequence as ordered pairs of the form (*term number, value*) on the coordinate plane. At left is the graph of the sequence from Example A.

Term number n	1	2	3	4	5	6	...	n
Value $f(n)$	7	2	-3	-8	-13	-18	...	$-5n + 12$

EXERCISES

For Exercises 1–3, find the function rule $f(n)$ for each sequence. Then find the 20th term in the sequence.

1.

n	1	2	3	4	5	6	…	n	…	20
$f(n)$	3	9	15	21	27	33	…		…	

2.

n	1	2	3	4	5	6	…	n	…	20
$f(n)$	1	-2	-5	-8	-11	-14	…		…	

3.

n	1	2	3	4	5	6	…	n	…	20
$f(n)$	-4	4	12	20	28	36	…		…	

For Exercises 4–6, find the rule for the nth figure. Then find the number of colored tiles or matchsticks in the 200th figure.

4.

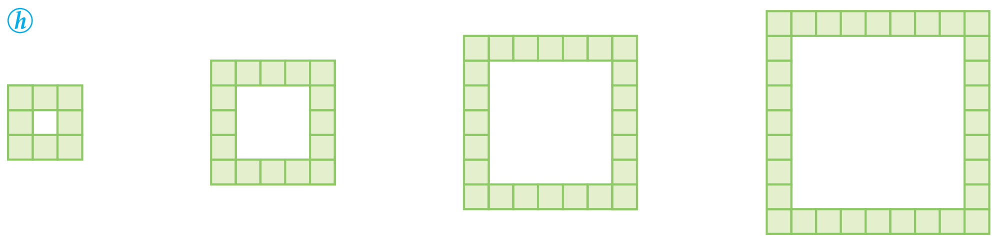


Figure number	1	2	3	4	5	6	…	n	…	200
Number of tiles	8						…		…	

5.

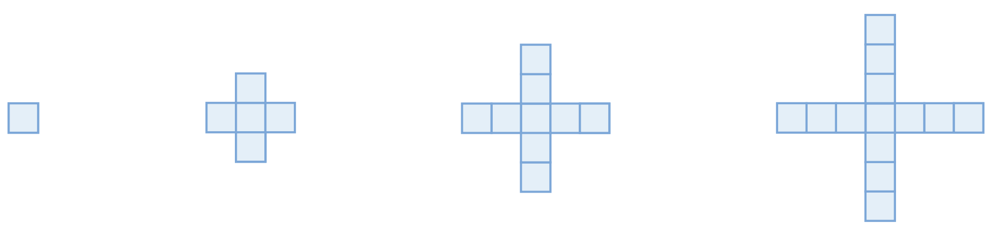


Figure number	1	2	3	4	5	6	…	n	…	200
Number of tiles		5					…		…	

6.

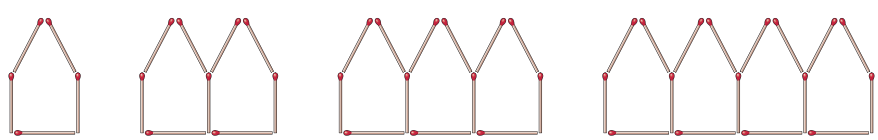
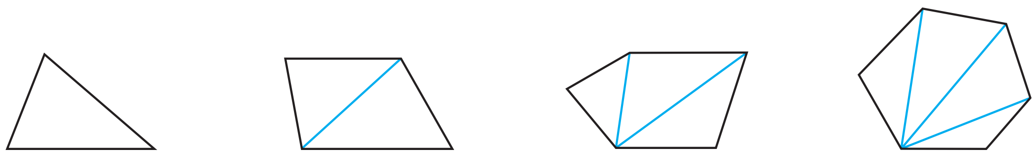


Figure number	1	2	3	4	5	6	…	n	…	200
Number of matchsticks	5	9					…		…	
Number of matchsticks in perimeter of figure	5	8					…		…	

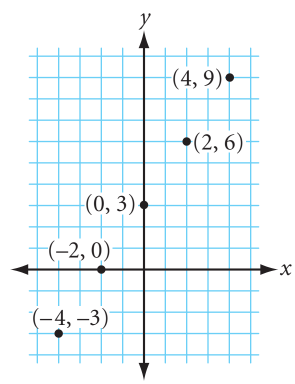
7. How many triangles are formed when you draw all the possible diagonals from just one vertex of a 35-gon? ⓗ



Number of sides	3	4	5	6	…	n	…	35
Number of triangles formed					…		…	

8. Graph the values in your tables from Exercises 4–6. Which set of points lies on a steeper line? What number in the rule gives a measure of steepness?

9. Find the rule for the set of points in the graph shown at right. Place the x-coordinate of each ordered pair in the top row of your table and the corresponding y-coordinate in the second row. What is the value of y in terms of x?



Review

For Exercises 10–13, sketch and carefully label the figure.

10. Equilateral triangle EQL with $\overline{QT}$ where T lies on $\overline{EL}$ and $\overline{QT} \perp \overline{EL}$

11. Isosceles obtuse triangle OLY with $\overline{OL} \cong \overline{YL}$ and angle bisector $\overline{LM}$

12. A cube with a plane passing through it; the cross section is rectangle $RECT$

13. A net for a rectangular solid with the dimensions 1 by 2 by 3 cm

14. Márisol's younger brother José was drawing triangles when he noticed that every triangle he drew turned out to have two sides congruent. José conjectures: "Look, Márisol, all triangles are isosceles." How should Márisol respond?

15. A midpoint divides a segment into two congruent segments. Point M divides segment $\overline{AY}$ into two congruent segments $\overline{AM}$ and $\overline{MY}$. What conclusion can you make? What type of reasoning did you use?

16. Tanya's favorite lunch is peanut butter and jelly on wheat bread with a glass of milk. Lately, she has been getting an allergic reaction after eating this lunch. She is wondering if she might be developing an allergy to peanut butter, wheat, or milk. What experiment could she do to find out which food it is? What type of reasoning would she be using?

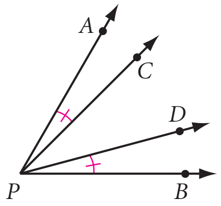
17. ***Mini-Investigation*** Do the geometry investigation and make a conjecture.

Given $\angle APB$ with points C and D in its interior and $m\angle APC = m\angle DPB$,

If $m\angle APD = 48°$, then $m\angle CPB = \underline{\ ?\ }$

If $m\angle CPB = 17°$, then $m\angle APD = \underline{\ ?\ }$

If $m\angle APD = 62°$, then $m\angle CPB = \underline{\ ?\ }$



Conjecture: If points C and D lie in the interior of $\angle APB$, and $m\angle APC = m\angle DPB$ then $m\angle APD = \underline{\ ?\ }$ (Overlapping angles property)

project

BEST-FIT LINES

The following table and graph show the mileage and lowest priced round-trip airfare between New York City and each destination city. Is there a relationship between the money you spend and how far you can travel?

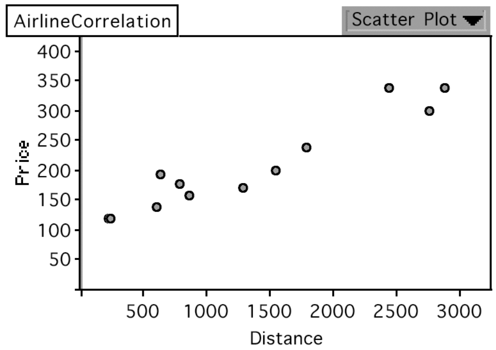
Fathom

With Fathom Dynamic Statistics™ software, you can plot your data points and find the linear equation that best fits your data.

Lowest Round-trip Airfares from New York City on February 25, 2002

Destination City	Distance (miles)	Price ($)
Boston	215	$118
Chicago	784	$178
Atlanta	865	$158
Miami	1286	$170
Denver	1791	$238
Phoenix	2431	$338
Los Angeles	2763	$298

Source: http://www.Expedia.com



Even though the data are not linear, you can find a linear equation that *approximately* fits the data. The graph of this equation is called the **line of best fit.** How would you use the line of best fit to predict the cost of a round-trip ticket to Seattle (2814 miles)? How would you use it to determine how far you could travel (in miles) with $250? How accurate do you think the answer would be?

Choose a topic and a relationship to explore. You can use data from the census (such as age and income), or data you collect yourself (such as number of ice cubes in a glass and melting time). For more sources and ideas, go to www.keymath.com/DG .

Collect data points. Use Fathom to graph your points and to find the line of best fit. Write a summary of your results.

LESSON

2.4 Mathematical Modeling

It's amazing what one can do when one doesn't know what one can't do.

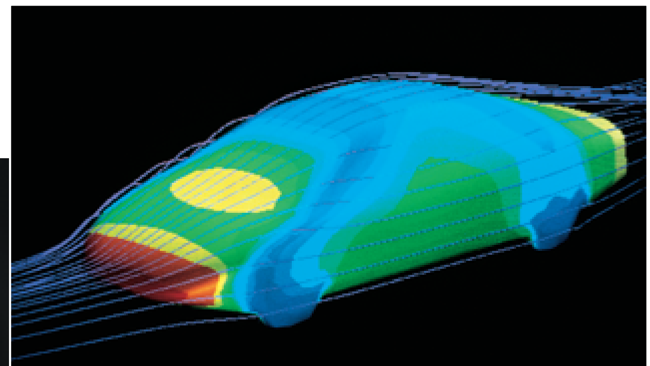
GARFIELD THE CAT

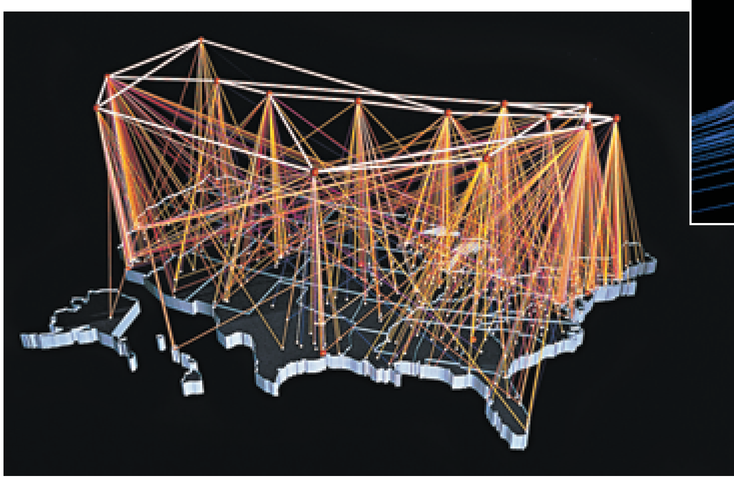
Physical models have many of the same features as the original object or activity they represent, but are often more convenient to study. For example, building a new airplane and testing it is difficult and expensive. But you can analyze a new airplane design by building a model and testing it in a wind tunnel.

In Chapter 1 you learned that geometry ideas such as points, lines, planes, triangles, polygons, and diagonals are **mathematical models** of physical objects.

When you draw graphs or pictures of situations, or when you write equations that describe a problem, you are creating mathematical models. A physical model of a complicated telecommunications network, for example, might not be practical, but you can draw a mathematical model of the network using points and lines.

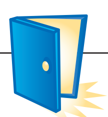
This computer model tests the effectiveness of the car's design for minimizing wind resistance.





This computer-generated model uses points and line segments to show the volume of data traveling to different locations on the National Science Foundation Network.

In this investigation, you will attempt to solve a problem first by acting it out, then by creating a mathematical model.



Investigation
Party Handshakes

Each of the 30 people at a party shook hands with everyone else. How many handshakes were there altogether?

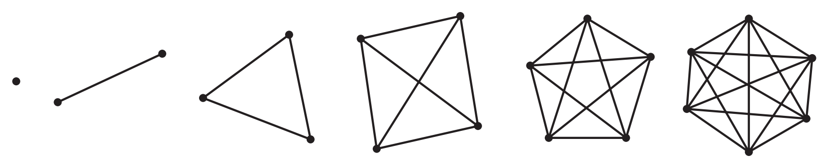
Step 1 Act out this problem with members of your group. Collect data for "parties" of one, two, three, and four people and record your results in a table.

People	1	2	3	4	…	30
Handshakes	0	1			…	

Step 2 | Look for a pattern. Can you generalize from your pattern to find the 30th term?

Acting out a problem is a powerful problem-solving strategy that can give you important insight into a solution. Were you able to make a generalization from just four terms? If so, how confident are you of your generalization? To collect more data, you can ask more classmates to join your group. You can see, however, that acting out a problem sometimes has its practical limitations. That's where you can use mathematical models.

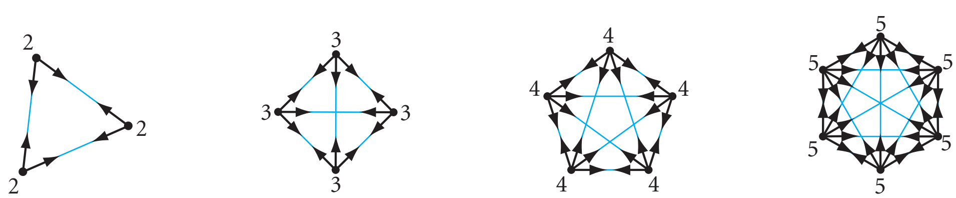
Step 3 | Model the problem by using points to represent people and line segments connecting the points to represent handshakes.



Record your results in a table like this one:

Number of points (people)	1	2	3	4	5	6	…	n	…	30
Number of segments (handshakes)	0	1					…		…	

Notice that the pattern does not have a constant difference. That is, the rule is not a linear function. So we need to look for a different kind of rule.



3 points
2 segments per vertex

4 points
3 segments per vertex

5 points
_?_ segments per vertex

6 points
_?_ segments per vertex

Step 4 | Refer to the table you made for Step 3. The pattern of differences is increasing by one: 1, 2, 3, 4, 5, 6, 7. Read the dialogue between Erin and Stephanie as they attempt to combine inductive and deductive reasoning to find the rule.

In the diagram with 3 vertices there are 2 segments from each vertex.

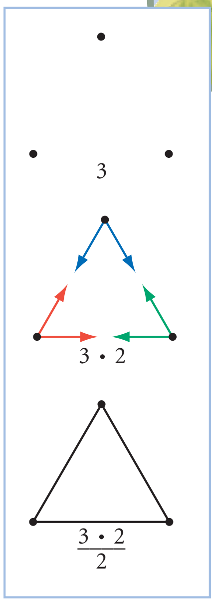
If there are 2 segments from each of the 3 vertices, why isn't the rule $2 \cdot 3$, or 6 segments?

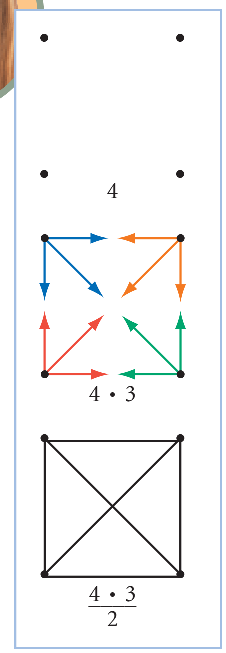
Because you are counting each segment twice, the answer is really $\frac{3 \cdot 2}{2}$, or 3 segments.

So, in the diagram with 4 vertices there are 3 segments from each vertex…

Right, but each segment got counted twice. So divide by 2.

…so there are $\frac{4 \cdot 3}{2}$, or 6 segments.





Let's continue with Stephanie and Erin's line of reasoning.

Step 5 | In the diagram with 5 vertices how many segments are there from each vertex? So the total number of segments written in factored form is $\frac{5 \cdot ?}{2}$.

Step 6 | Complete the table below by expressing the total number of segments in factored form.

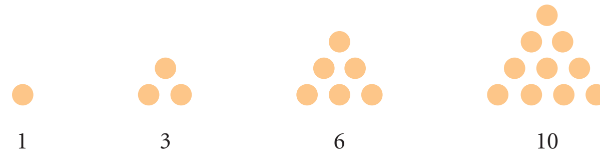
Number of points (people)	1	2	3	4	5	6	…	n
Number of segments (handshakes)	$\frac{(1)(0)}{2}$	$\frac{(2)(1)}{2}$	$\frac{(3)(2)}{2}$	$\frac{(4)(3)}{2}$	$\frac{(5)(?)}{2}$	$\frac{(6)(?)}{2}$	…	$\frac{(?)(?)}{2}$

Step 7 | The larger of the two factors in the numerator represents the number of points. What does the smaller of the two numbers in the numerator represent? Why do we divide by 2?

Step 8 | Write a function rule. How many handshakes were there at the party?

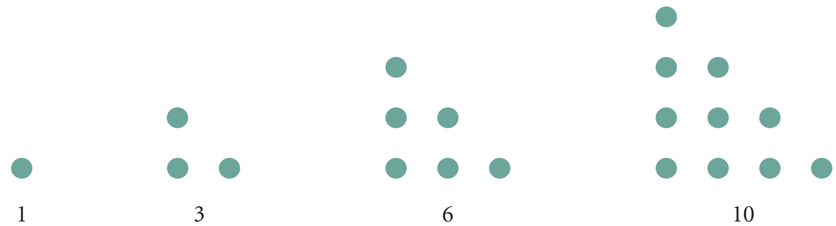
Fifteen pool balls can be arranged in a triangle, so 15 is a triangular number.

The numbers in the pattern in the previous investigation are called the **triangular numbers** because you can arrange them into a triangular pattern of dots.



The triangular numbers appear in many geometric situations. You will see some of them in the exercises.

Here is a visual approach to arrive at the rule for this special pattern of numbers. If we arrange the triangular numbers in stacks,



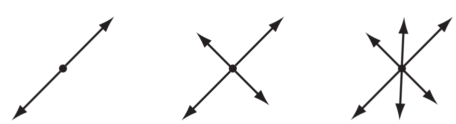
you can see that each is half of a **rectangular number.**

To get the total number of dots in a rectangular array, you multiply the number of rows by the number of dots in each row. In the case of this rectangular array, the nth rectangle has $n(n+1)$ dots. So, the triangular array has $\frac{n(n+1)}{2}$ dots.

EXERCISES

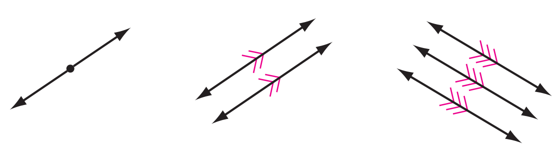
For Exercises 1–6, draw the next figure. Complete a table and find the function rule. Then find the 35th term.

1. Lines passing through the same point are **concurrent.** Into how many regions do 35 concurrent lines divide the plane?

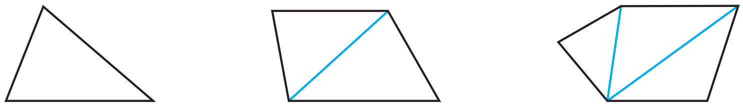


Lines	1	2	3	4	5	…	n	…	35
Regions	2					…		…	

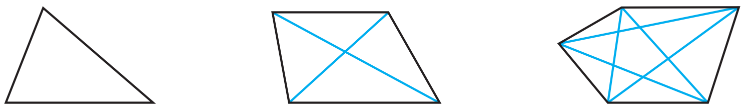
2. Into how many regions do 35 parallel lines in a plane divide that plane?



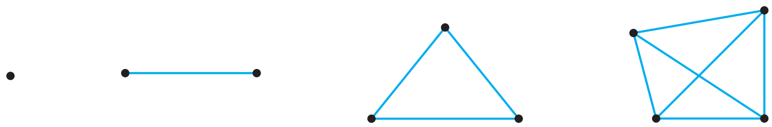
3. How many diagonals can you draw from one vertex in a polygon with 35 sides?



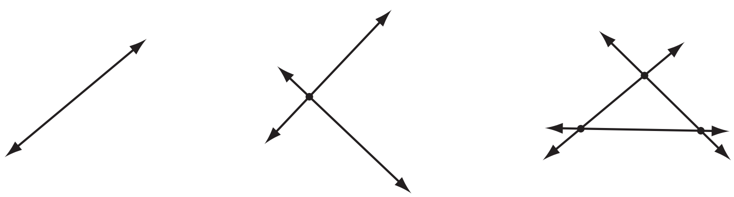
4. What's the total number of diagonals in a 35-sided polygon? ⓗ



5. If you place 35 points on a piece of paper so that no three points are in a line, how many line segments are necessary to connect each point to all the others? ⓗ



6. If you draw 35 lines on a piece of paper so that no two lines are parallel to each other and no three lines are concurrent, how many times will they intersect? ⓗ



7. Look at the formulas you found in Exercises 4–6. Describe how the formulas are related. Then explain how the three problems are related geometrically. ⓗ

For Exercises 8–10, draw a diagram, find the appropriate geometric model, and solve.

8. If 40 houses in a community all need direct lines to one another in order to have telephone service, how many lines are necessary? Is that practical? Sketch and describe two models: first, model the situation in which direct lines connect every house to every other house and, second, model a more practical alternative.

9. If each team in a ten-team league plays each of the other teams four times in a season, how many league games are played during one season? What geometric figures can you use to model teams and games played? ⓗ

10. Each person at a party shook hands with everyone else exactly once. There were 66 handshakes. How many people were at the party? ⓗ

Review

For Exercises 11–19, identify the statement as true or false. For each false statement, explain why it is false or sketch a counterexample.

11. The largest chord of a circle is a diameter of the circle.

12. The vertex of $\angle TOP$ is point O.

13. An isosceles right triangle is a triangle with an angle measuring 90° and no two sides congruent.

14. If $\overleftrightarrow{AB}$ intersects $\overleftrightarrow{CD}$ in point E, then $\angle AED$ and $\angle BED$ form a linear pair of angles. ⓗ

15. If two lines lie in the same plane and are perpendicular to the same line, they are perpendicular.

16. The opposite sides of a kite are never parallel.

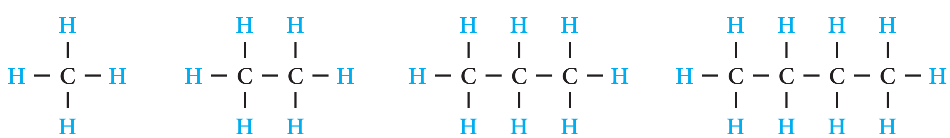
17. A rectangle is a parallelogram with all sides congruent.

18. A line segment that connects any two vertices in a polygon is called a diagonal.

19. To show that two lines are parallel, you mark them with the same number of arrowheads.

20. Hydrocarbons are molecules that consist of carbon (C) and hydrogen (H). Hydrocarbons in which all the bonds between the carbon atoms are single bonds are called *alkanes*. The first four alkanes are modeled below.

Sketch the alkane with eight carbons in the chain. What is the general rule for alkanes $(C_nH_?)$? In other words, if there are n carbon atoms (C), how many hydrogen atoms (H) are in the alkane?



Methane (CH_4) | Ethane (C_2H_6) | Propane (C_3H_8) | Butane (C_4H_{10})

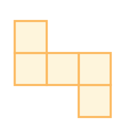
Science CONNECTION

Organic chemistry is the study of carbon compounds and their reactions. Drugs, vitamins, synthetic fibers, and food all contain organic molecules. Organic chemists continue to improve our quality of life by the advances they make in medicine, nutrition, and manufacturing. To learn about new advances in organic chemistry, go to www.keymath.com/DG .

IMPROVING YOUR VISUAL THINKING SKILLS

Pentominoes II

In Pentominoes I, you found the 12 pentominoes. Which of the 12 pentominoes can you cut along the edges and fold into a box without a lid? Here is an example.

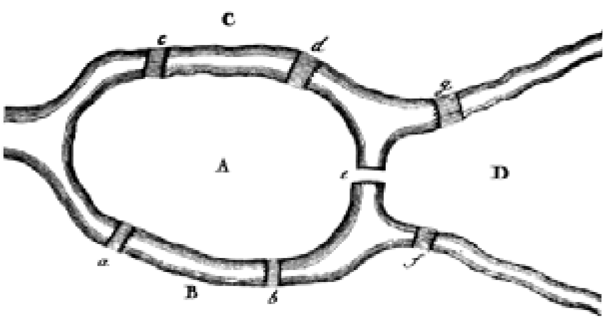


Exploration

The Seven Bridges of Königsberg

Leonhard Euler

The River Pregel runs through the university town of Königsberg (now Kaliningrad in Russia). In the middle of the river are two islands connected to each other and to the rest of the city by seven bridges. Many years ago, a tradition developed among the townspeople of Königsberg. They challenged one another to make a round trip over all seven bridges, walking over each bridge once and only once before returning to the starting point.



The seven bridges of Königsberg

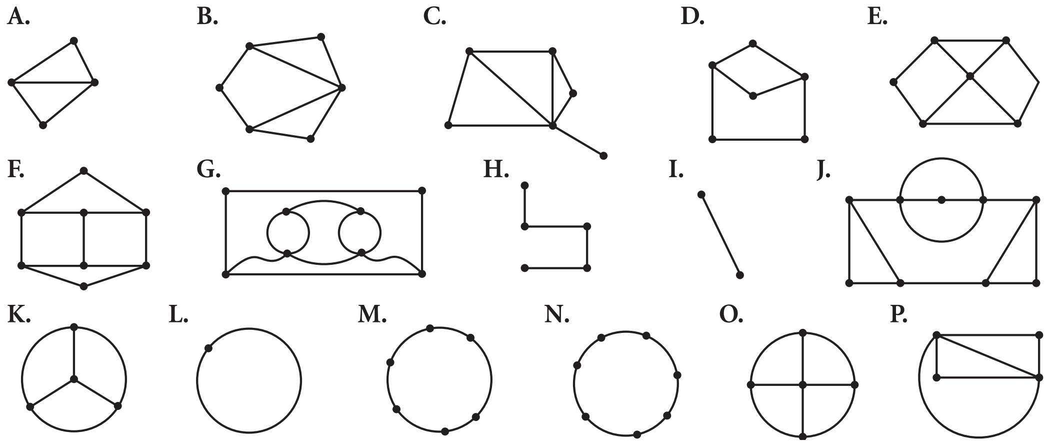
For a long time no one was able to do it, and yet no one was able to show that it couldn't be done. In 1735, they finally wrote to Leonhard Euler (1707–1783), a Swiss mathematician, asking for his help on the problem. Euler (pronounced "oyler") reduced the problem to a network of paths connecting the two sides of the rivers C and B, and the two islands A and D, as shown in the network at right. Then Euler demonstrated that the task is impossible.

In this activity you will work with a variety of networks to see if you can come up with a rule to find out whether a network can or cannot be "traveled."

Activity
Traveling Networks

A collection of points connected by paths is called a **network.** When we say a network can be traveled, we mean that the network can be drawn with a pencil without lifting the pencil off the paper and without retracing any paths. (Points can be passed over more than once.)

Step 1 | Try these networks and see which ones can be traveled and which are impossible to travel.

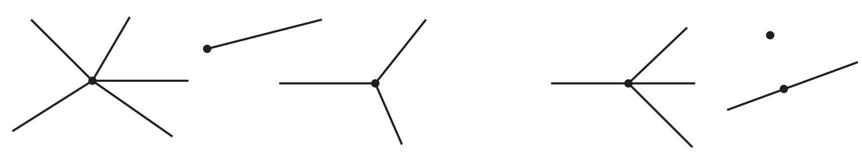


Which networks were impossible to travel? Are they impossible or just difficult? How can you be sure? As you do the next few steps, see if you can find the reason why some networks are impossible to travel.

Step 2 | Draw the River Pregel and the two islands shown on the first page of this exploration. Draw an eighth bridge so that you can travel over all the bridges exactly once if you start at point *C* and end at point *B*.

Step 3 | Draw the River Pregel and the two islands. Can you draw an eighth bridge so that you can travel over all the bridges exactly once, starting and finishing at the same point? How many solutions can you find?

Step 4 | Euler realized that it is the points of intersection that determine whether a network can be traveled. Each point of intersection is either "odd" or "even."



Odd points　　　　Even points

Did you find any networks that have only one odd point? Can you draw one? Try it. How about three odd points? Or five odd points? Can you create a network that has an odd number of odd points? Explain why or why not.

Step 5 | How does the number of even points and odd points affect whether a network can be traveled?

Conjecture

A network can be traveled if __?__.

LESSON

2.5

Angle Relationships

Discovery consists of looking at the same thing as everyone else and thinking something different.

ALBERT SZENT-GYÖRGYI

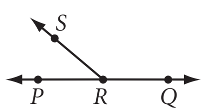
Now that you've had experience with inductive reasoning, let's use it to start discovering geometric relationships. This investigation is the first of many investigations you will do using your geometry tools.

Create an investigation section in your notebook. Include a title and illustration for each investigation and write a statement summarizing the results of each one.

Investigation 1
The Linear Pair Conjecture

You will need

- a protractor



Step 1 On a sheet of paper, draw $\overleftrightarrow{PQ}$ and place a point R between P and Q. Choose another point S not on $\overleftrightarrow{PQ}$ and draw $\overrightarrow{RS}$. You have just created a linear pair of angles. Place the "zero edge" of your protractor along $\overleftrightarrow{PQ}$. What do you notice about the sum of the measures of the linear pair of angles?

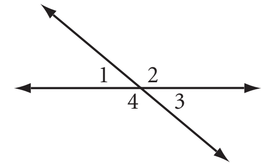
Step 2 Compare your results with those of your group. Does everyone make the same observation? Complete the statement.

Linear Pair Conjecture C-1

If two angles form a linear pair, then $\underline{\ ?\ }$.

The important conjectures have been given a name and a number. Start a list of them in your notebook. The Linear Pair Conjecture (C-1) and the Vertical Angles Conjecture (C-2) should be the first entries on your list. Make a sketch for each conjecture.

In the previous investigation you discovered the relationship between a linear pair of angles, such as ∠1 and ∠2 in the diagram at right.

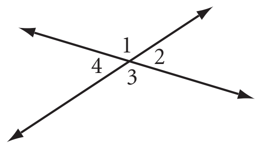


You will discover the relationship between vertical angles, such as ∠1 and ∠3, in the next investigation.

Investigation 2
Vertical Angles Conjecture

You will need

- a protractor
- patty paper



Step 1 Draw two intersecting lines onto patty paper or tracing paper. Label the angles as shown. Which angles are vertical angles?

Step 2 Fold the paper so that the vertical angles lie over each other. What do you notice about their measures?

Step 3 Repeat this investigation with another pair of intersecting lines.

Step 4 Compare your results with the results of others. Complete the statement.

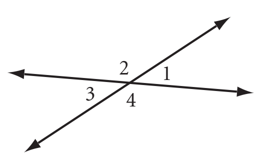
Vertical Angles Conjecture C-2

If two angles are vertical angles, then $\underline{\ ?\ }$.

You used inductive reasoning to discover both the Linear Pair Conjecture and the Vertical Angles Conjecture. Are they related in any way? That is, if we accept the Linear Pair Conjecture as true, can we use deductive reasoning to show that the Vertical Angles Conjecture must be true?

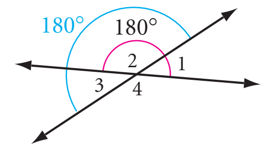
EXAMPLE

The Linear Pair Conjecture states that every linear pair adds up to 180°. Using this conjecture and the diagram, write a logical argument explaining why ∠1 must be congruent to ∠3.



▶ Solution

You can see that the measures of $\angle 1$ and $\angle 2$ add up to 180°, and that the measures of $\angle 3$ and $\angle 2$ also add up to 180°. Using algebra, we can write a logical argument to show that $\angle 1$ and $\angle 3$ must be congruent.



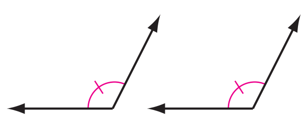
According to the Linear Pair Conjecture, $m\angle 1 + m\angle 2 = 180°$ and $m\angle 2 + m\angle 3 = 180°$. By substituting $m\angle 2 + m\angle 3$ for 180° in the first statement, you get $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$. By the subtraction property of equality, you can subtract $m\angle 2$ from both sides of the equation to get $m\angle 1 = m\angle 3$. Therefore, vertical angles 1 and 3 have equal measures and are congruent.

Here are the algebraic steps:

$m\angle 2 + m\angle 3 = 180°$
$m\angle 1 + m\angle 2 = 180°$
$m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$
thus $m\angle 1 = m\angle 3$
therefore $\angle 1 \cong \angle 3$

This type of logical explanation, written as a paragraph, is called a **paragraph proof.**

Now consider another idea. You discovered the Vertical Angles Conjecture: If two angles are vertical angles, then they are congruent. Does that also mean that all congruent angles are vertical angles? The **converse** of an “if-then” statement switches the “if” and “then” parts. The converse of the Vertical Angles Conjecture may be stated: If two angles are congruent, then they are vertical angles. Is this converse statement true? Remember that if you can find even one counterexample, like the diagram below, then the statement is false.



Therefore, the converse of the Vertical Angles Conjecture is false.

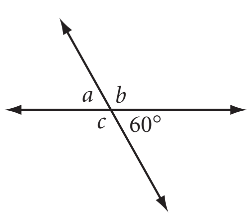
EXERCISES

You will need

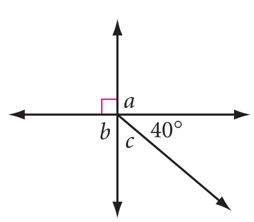
Geometry software for Exercise **12**

Without using a protractor, but with the aid of your two new conjectures, find the measure of each lettered angle in Exercises 1–5. Copy the diagrams so that you can write on them. List your answers in alphabetical order.

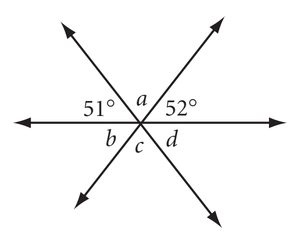
1.



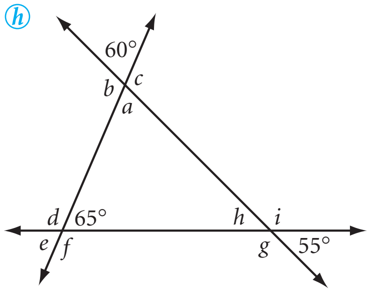
2.



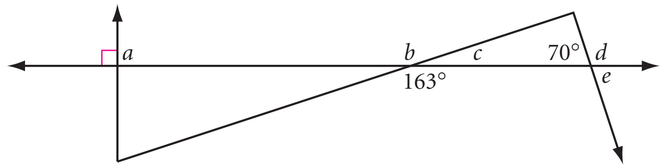
3.

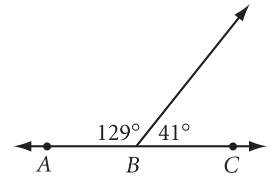


4. ⓗ



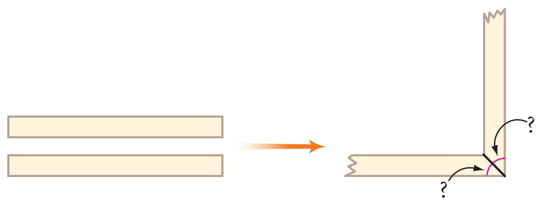
5.

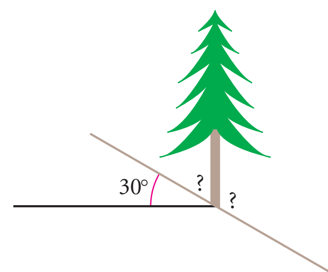




6. Points *A*, *B*, and *C* at right are collinear. What's wrong with this picture?

7. Yoshi is building a cold frame for his plants. He wants to cut two wood strips so that they'll fit together to make a right-angled corner. At what angle should he cut ends of the strips?





8. A tree on a 30° slope grows straight up. What are the measures of the greatest and smallest angles the tree makes with the hill? Explain.

9. You discovered that if a pair of angles is a linear pair then the angles are supplementary. Does that mean that all supplementary angles form a linear pair of angles? Is the converse true? If not, sketch a counterexample.

10. If two congruent angles are supplementary, what must be true of the two angles? Make a sketch, then complete the following conjecture: If two angles are both congruent and supplementary, then __?__.

11. Using algebra, write a paragraph proof that explains why the conjecture from Exercise 10 is true.

12. *Technology* Use geometry software to construct two intersecting lines. Measure a pair of vertical angles. Use **calculate** to find the ratio of their measures. What is the ratio? Drag one of the lines. Does the ratio ever change? Does this demonstration convince you that the Vertical Angles Conjecture is true? Does it explain why it is true?

Review

For Exercises 13–17, sketch, label, and mark the figure.

13. Scalene obtuse triangle *PAT* with $PA = 3$ cm, $AT = 5$ cm, and $\angle A$ an obtuse angle

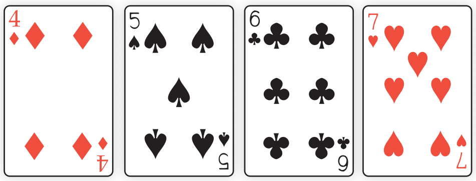
14. A quadrilateral that has rotational symmetry but not reflectional symmetry

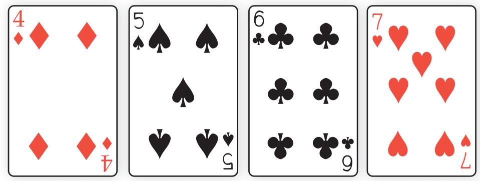
15. A circle with center at *O* and radii $\overline{OA}$ and $\overline{OT}$ creating a minor arc $\overset{\frown}{AT}$

16. A pyramid with an octagonal base

17. A 3-by-4-by-6-inch rectangular solid rests on its smallest face. Draw lines on the three visible faces, showing how you can divide it into 72 identical smaller cubes.

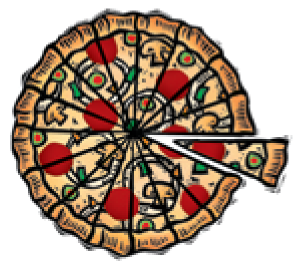
18. Miriam the Magnificent placed four cards face up (the first four cards shown below). Blindfolded, she asked someone from her audience to come up to the stage and turn one card 180°.


Before turn

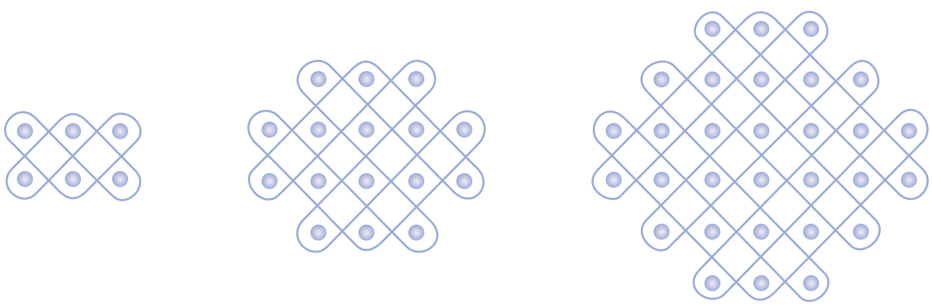

After turn

Miriam removed her blindfold and claimed she was able to determine which card was turned 180°. What is her trick? Can you figure out which card was turned? Explain.

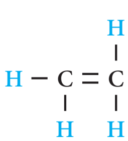
19. If a pizza is cut into 16 congruent pieces, how many degrees are in each angle at the center of the pizza?

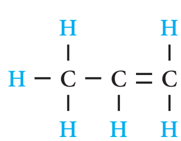


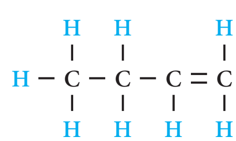
20. Paulus Gerdes, a mathematician from Mozambique, uses traditional *lusona* patterns from Angola to practice inductive thinking. Shown below are three *sona* designs. Sketch the fourth *sona* design, assuming the pattern continues.



21. Hydrocarbon molecules in which all the bonds between the carbon atoms are single bonds except one double bond are called *alkenes*. The first three alkenes are modeled below.

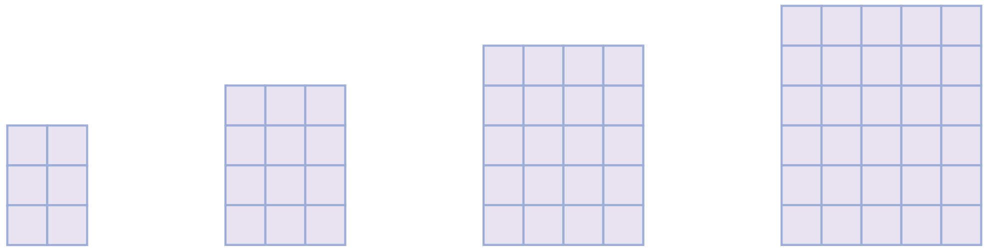

Ethene
(C_2H_4)


Propene
(C_3H_6)


Butene
(C_4H_8)

Sketch the alkene with eight carbons in the chain. What is the general rule for alkenes $(C_nH_?)$? In other words, if there are n carbon atoms (C), how many hydrogen atoms (H) are in the alkene?

22. If the pattern of rectangles continues, what is the rule for the perimeter of the nth rectangle, and what is the perimeter of the 200th rectangle?



Perimeter in a rectangular pattern

Rectangle	1	2	3	4	5	6	...	n	...	200
Perimeter of rectangle	10	14	18				...		...	

23. The twelfth grade class of 80 students is assembled in a large circle on the football field at halftime. Each student is connected by a string to each of the other class members. How many pieces of string are necessary to connect each student to all the others? ⓗ

24. If you draw 80 lines on a piece of paper so that no 2 lines are parallel to each other and no 3 lines pass through the same point, how many intersections will there be? ⓗ

25. If there are 20 couples at a party, how many different handshakes can there be between pairs of people? Assume that the two people in each couple do not shake hands with each other. ⓗ

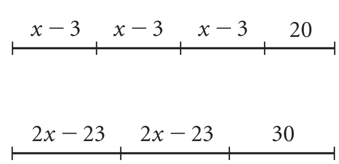
26. If a polygon has 24 sides, how many diagonals are there from each vertex? How many diagonals are there in all?

27. If a polygon has a total of 560 diagonals, how many vertices does it have? ⓗ

IMPROVING YOUR ALGEBRA SKILLS

Number Line Diagrams

1. The two segments at right have the same length. Translate the number line diagram into an equation, then solve for the variable x.



2. Translate this equation into a number line diagram.
$2(x + 3) + 14 = 3(x - 4) + 11$

LESSON

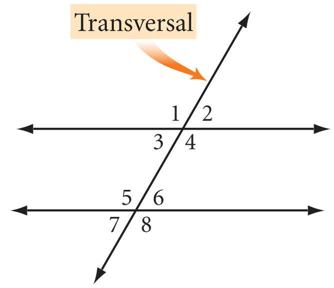
2.6

Special Angles on Parallel Lines

The greatest mistake you can make in life is to be continually fearing that you will make one.

ELLEN HUBBARD

A line intersecting two or more other lines in the plane is called a **transversal.** A transversal creates different types of angle pairs. Three types are listed below.

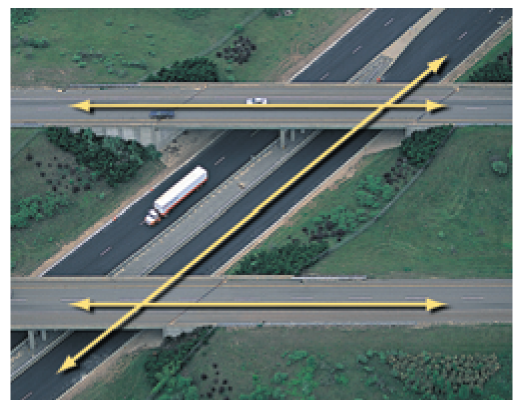


One pair of **corresponding angles** is $\angle 1$ and $\angle 5$. Can you find three more pairs of corresponding angles?

One pair of **alternate interior angles** is $\angle 3$ and $\angle 6$. Do you see another pair of alternate interior angles?

One pair of **alternate exterior angles** is $\angle 2$ and $\angle 7$. Do you see the other pair of alternate exterior angles?

When parallel lines are cut by a transversal, there is a special relationship among the angles. Let's investigate.

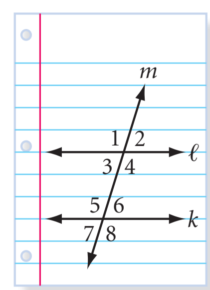


Investigation 1
Which Angles Are Congruent?

You will need

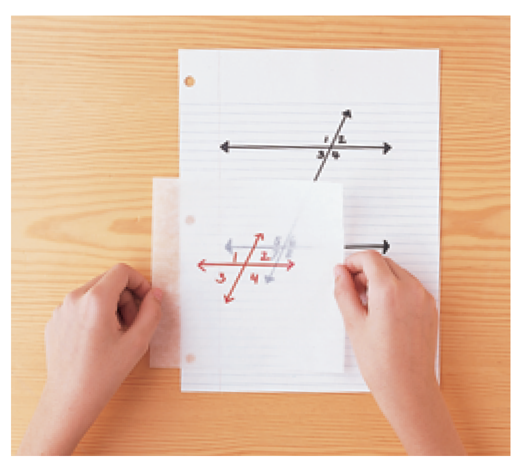
- lined paper or a straightedge
- patty paper
- a protractor

Using the lines on your paper as a guide, draw a pair of parallel lines. Or use both edges of your ruler or straightedge to create parallel lines. Label them k and ℓ. Now draw a transversal that intersects the parallel lines. Label the transversal m, and label the angles with numbers, as shown at right.



Step 1 Place a piece of patty paper over the set of angles 1, 2, 3, and 4. Copy the two intersecting lines m and ℓ and the four angles onto the patty paper.

Slide the patty paper down to the intersection of lines m and k, and compare angles 1 through 4 with each of the corresponding angles 5 through 8. What is the relationship between corresponding angles? Alternate interior angles? Alternate exterior angles?

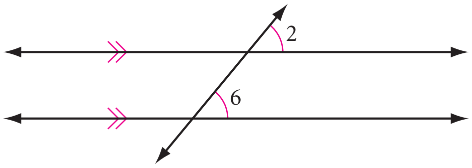


Compare your results with the results of others in your group and complete the three conjectures below.

C-3a

Corresponding Angles Conjecture, or CA Conjecture

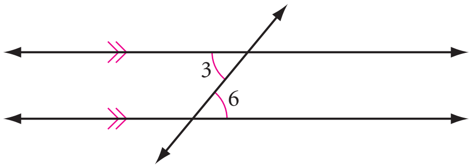
If two parallel lines are cut by a transversal, then corresponding angles are ?.



C-3b

Alternate Interior Angles Conjecture, or AIA Conjecture

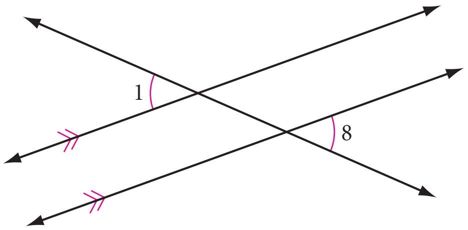
If two parallel lines are cut by a transversal, then alternate interior angles are ?.



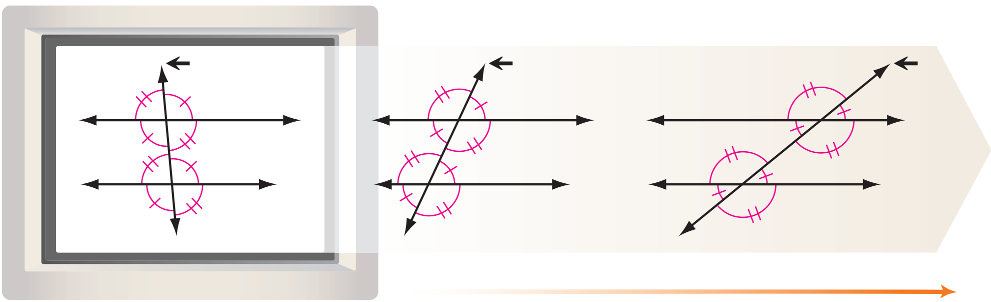
C-3c

Alternate Exterior Angles Conjecture, or AEA Conjecture

If two parallel lines are cut by a transversal, then alternate exterior angles are ?.



For an interactive version of this sketch, visit www.keymath.com/DG .



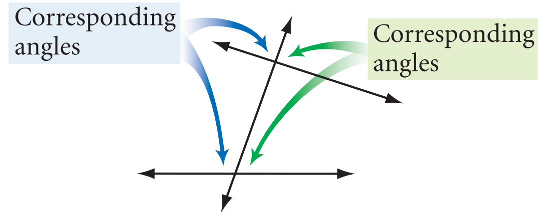
The three conjectures you wrote can all be combined to create a Parallel Lines Conjecture, which is really three conjectures in one.

C-3

Parallel Lines Conjecture

If two parallel lines are cut by a transversal, then corresponding angles are ?, alternate interior angles are ?, and alternate exterior angles are ?.

Step 2 What happens if the lines you start with are not parallel? Check whether your conjectures will work with nonparallel lines.

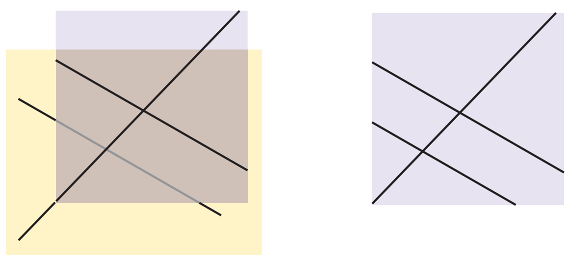


What about the converse of each of your conjectures? Suppose you know that a pair of corresponding angles, or alternate interior angles, is congruent. Will the lines be parallel? Is it possible for the angles to be congruent but for the lines not to be parallel?

Investigation 2
Is the Converse True?

You will need

- lined paper or a straightedge
- patty paper
- a protractor

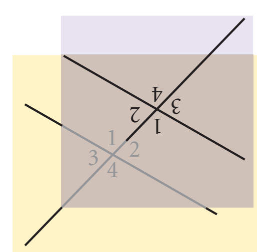


Step 1 Draw two intersecting lines on your paper. Copy these lines onto a piece of patty paper. Because you copied the angles, the two sets of angles are congruent.

Slide the top copy so that the transversal stays lined up.

Trace the lines and the angles from the bottom original onto the patty paper again. When you do this, you are constructing sets of congruent corresponding angles. Mark the congruent angles.

Are the two lines parallel? You can test to see if the distance between the two lines remains the same, which guarantees that they will never meet.



Step 2 Repeat Step 1, but this time rotate your patty paper 180° so that the transversal lines up again. What kinds of congruent angles have you created? Trace the lines and angles and mark the congruent angles. Are the lines parallel? Check them.

Step 3 | Compare your results with those of your group. If your results do not agree, discuss them until you have convinced each other. Complete the conjecture below and add it to your conjecture list.

Converse of the Parallel Lines Conjecture

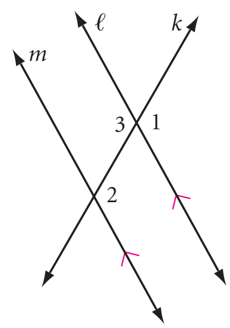
C-4

If two lines are cut by a transversal to form pairs of congruent corresponding angles, congruent alternate interior angles, or congruent alternate exterior angles, then the lines are __?__.

You used inductive reasoning to discover all three parts of the Parallel Lines Conjecture. However, if you accept any one of them as true, you can use deductive reasoning to show that the others are true.

EXAMPLE

Suppose we assume that the Vertical Angles Conjecture is true. Write a paragraph proof showing that if corresponding angles are congruent, then the Alternate Interior Angles Conjecture is true.



▶ Solution

Paragraph Proof

Lines ℓ and m are parallel and intersected by transversal k. Pick any two alternate interior angles, such as $\angle 2$ and $\angle 3$. According to the Corresponding Angles Conjecture, $\angle 2 \cong \angle 1$. And, according to the Vertical Angles Conjecture, $\angle 1 \cong \angle 3$. Substitute $\angle 3$ for $\angle 1$ in the first statement to get $\angle 2 \cong \angle 3$. But $\angle 2$ and $\angle 3$ are alternate interior angles. Therefore, if the corresponding angles are congruent, then the alternate interior angles are congruent. ■

Here are the algebraic steps:

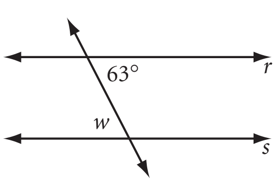
$$\angle 2 \cong \angle 1$$
$$\angle 3 \cong \angle 1$$
$$\text{So} \quad \angle 2 \cong \angle 3$$

It helps to visualize each statement and to mark all congruences you know on your paper.

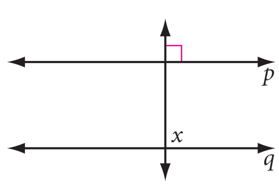
EXERCISES

Use your new conjectures in Exercises 1–6. A small letter in an angle represents the angle measure.

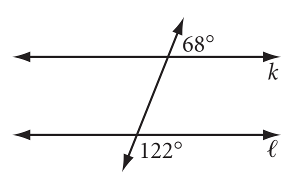
1. $r \parallel s$
$w = \underline{?}$



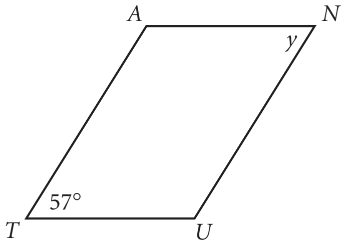
2. $p \parallel q$
$x = \underline{?}$



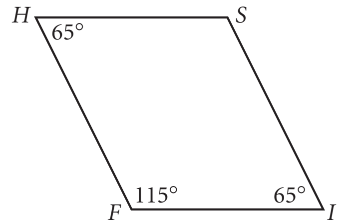
3. Is line k parallel to line ℓ?



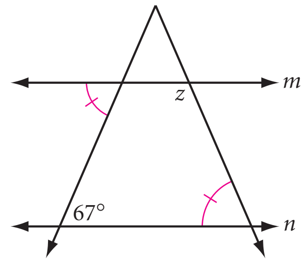
4. Quadrilateral *TUNA* is a parallelogram.
$y = \underline{\ ?\ }$ ⓗ



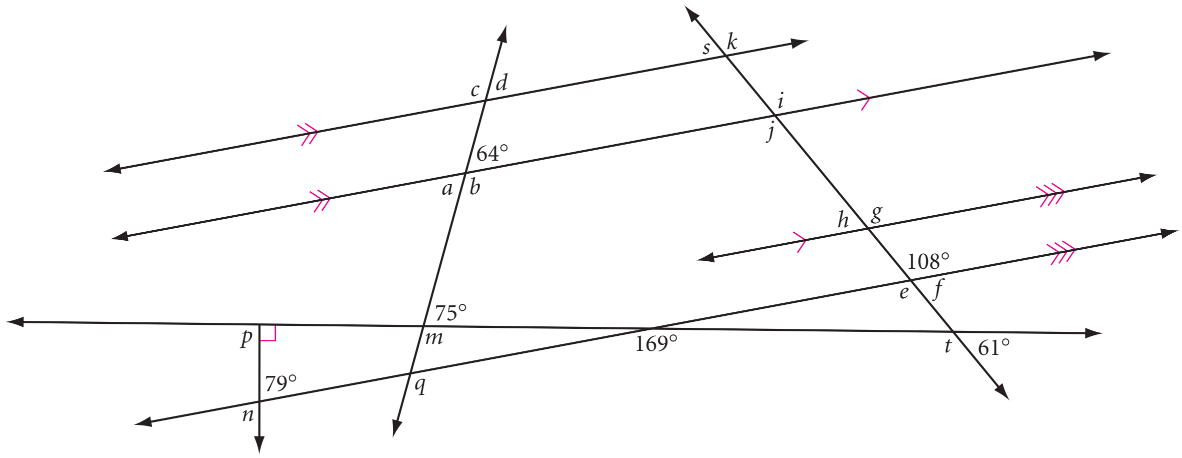
5. Is quadrilateral *FISH* a parallelogram?



6. $m \parallel n$
$z = \underline{\ ?\ }$ ⓗ

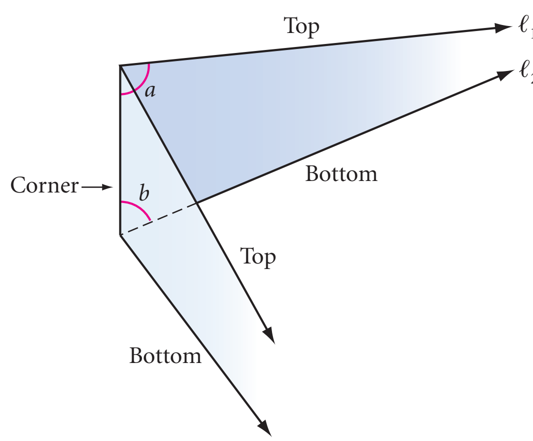


7. Trace the diagram below. Calculate each lettered angle measure. ⓗ

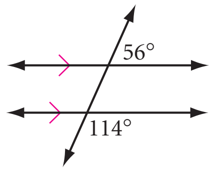


8. You've seen before how parallel lines appear to meet in the distance. Let's look at the converse of this effect: The top and the bottom of the Vietnam Veterans Memorial Wall appear to be parallel because they appear to meet so far in the distance. Consider the diagram of the corner of the memorial, shown below. You know that line ℓ_1 and line ℓ_2 eventually meet. Is the blue shaded portion of the wall a rectangle? Write a paragraph proof explaining why it is or is not a rectangle.

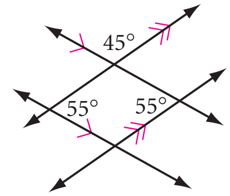
Sculptor Maya Lin designed the Vietnam Veterans Memorial Wall in Washington, D.C. Engraved in the granite wall are the names of United States armed forces service members who died in the Vietnam War or remain missing in action. To learn more about the Memorial Wall and Lin's other projects, visit www.keymath.com/DG .



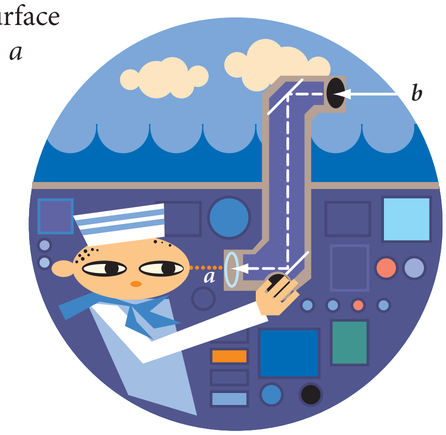
9. What's wrong with this picture?



10. What's wrong with this picture?



11. A periscope permits a sailor on a submarine to see above the surface of the ocean. This periscope is designed so that the line of sight a is parallel to the light ray b. The middle tube is perpendicular to the top and bottom tubes. What are the measures of the incoming and outgoing angles formed by the light rays and the mirrors in this periscope? Are the surfaces of the mirrors parallel? How do you know?



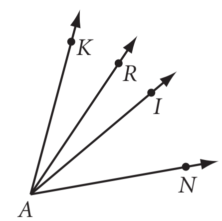
Review

12. What type (or types) of triangle has one or more lines of symmetry?

13. What type (or types) of quadrilateral has only rotational symmetry? ⓗ

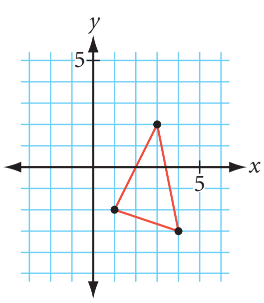
14. If D is the midpoint of $\overline{AC}$ and C is the midpoint of $\overline{BD}$, what is the length of $\overline{AB}$ if $BD = 12$ cm?

15. If $\overrightarrow{AI}$ is the angle bisector of $\angle KAN$ and $\overrightarrow{AR}$ is the angle bisector of $\angle KAI$, what is $m\angle RAN$ if $m\angle RAK = 13°$?

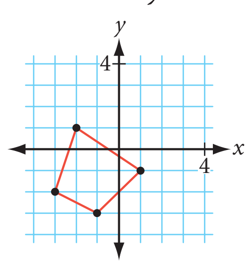


For Exercises 16–18, draw each polygon on graph paper. Relocate the vertices according to the rule. Connect the new points to form a new polygon. Describe what happened to the figure. Is the new polygon congruent to the original?

16. **Rule:** Subtract 1 from each x-coordinate. ⓗ



17. **Rule:** Reverse the sign of each x- and y-coordinate.

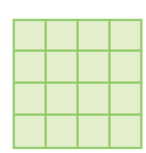


18. **Rule:** Switch the x- and y-coordinates. Pentagon $LEMON$ with vertices:

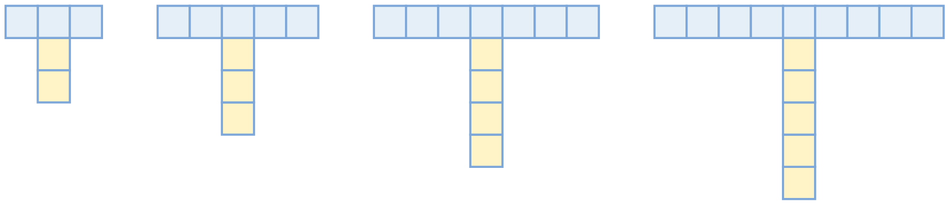
$L(-4, 2)$
$E(-4, -3)$
$M(0, -5)$
$O(3, 1)$
$N(-1, 4)$

19. If everyone in the town of Skunk's Crossing (population 84) has a telephone, how many different lines are needed to connect all the phones to each other?

20. How many squares of all sizes are in a 4-by-4 grid of squares? (There are more than 16!) ⓗ



21. Assume the pattern of blue and yellow shaded **T**'s continues. Copy and complete the table for blue shaded and yellow shaded squares and for the total number of squares. Ⓗ

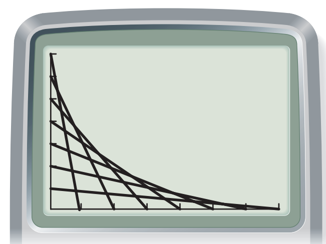


The T-formation

Figure number	1	2	3	4	5	6	…	n	…	35
Number of yellow squares	2						…		…	
Number of blue squares	3						…		…	
Total number of squares	5						…		…	

project

LINE DESIGNS



Can you use your graphing calculator to make the line design shown at right? You'll need to recall some algebra. Here are some hints.

1. The x- and y-ranges are set to minimums of 0 and maximums of 7.
2. The design consists of the graphs of seven lines.
3. The equation for one of the lines is $y = -\frac{1}{7}x + 1$.
4. There's a simple pattern in the slopes and y-intercepts of the lines.

You're on your own from here. Experiment! Then create a line design of your own and write the equations for it.

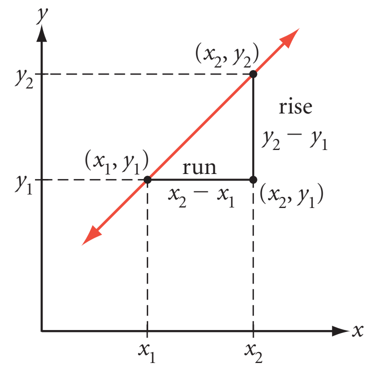
Slope

The slope of a line is a measure of its steepness. Measuring slope tells us the steepness of a hill, the pitch of a roof, or the incline of a ramp. On a graph, slope can tell us the rate of change, or speed.

To calculate slope, you find the ratio of the vertical distance to the horizontal distance traveled, sometimes referred to as "rise over run."

$$\text{slope} = \frac{\text{vertical change}}{\text{horizontal change}}$$

One way to find slope is to use a **slope triangle.** Then use the coordinates of its vertices in the formula.



Slope Formula

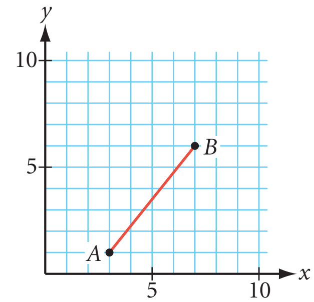
The slope m of a line (or segment) through two points with coordinates (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where $x_2 - x_1 \neq 0$.

EXAMPLE

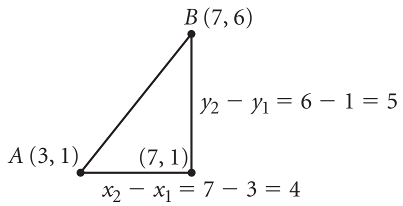
Draw the slope triangle and find the slope for $\overline{AB}$.



Solution

Draw the horizontal and vertical sides of the slope triangle below the line. Use them to calculate the side lengths.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 1}{7 - 3} = \frac{5}{4}$$



Note that if the slope triangle is *above* the line, you subtract the numbers in reverse order, but still get the same result.

$$m = \frac{1 - 6}{3 - 7} = \frac{-5}{-4} = \frac{5}{4}$$

The slope is positive when the line goes up from left to right. The slope is negative when the line goes down from left to right. When is the slope 0? What is the slope of a vertical line?

EXERCISES

In Exercises 1–3 find the slope of the line through the given points.

1. (16, 0) and (12, 8) **2.** (−3, −4) and (−16, 8) **3.** (5.3, 8.2) and (0.7, −1.5)

4. A line through points (−5, 2) and (2, *y*) has a slope of 3. Find *y*.

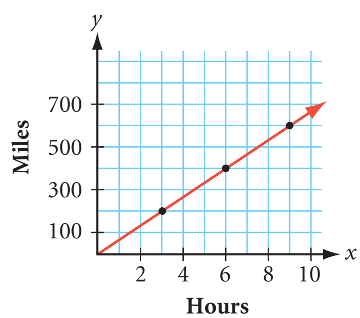
5. A line through points (*x*, 2) and (7, 9) has a slope of $\frac{7}{3}$. Find *x*.

6. Find the coordinates of three more points that lie on the line passing through the points (0, 0) and (3, −4). Explain your method.

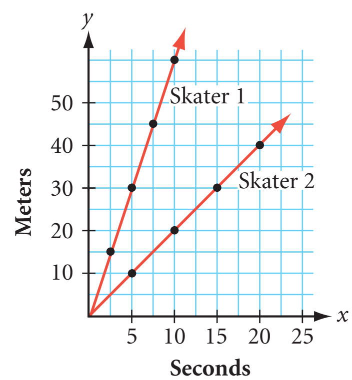
7. What is the speed, in miles per hour, represented by Graph A?

8. From Graph B, which in-line skater is faster? How much faster?

9. The grade of a road is its slope, given as a percent. For example, a road with a 6% grade has slope $\frac{6}{100}$. It rises 6 feet for every 100 feet of horizontal run. Describe a 100% grade. Do you think you could drive up it? Could you walk up it? Is it possible for a grade to be greater than 100%?



Graph A



Graph B

10. What's the slope of the roof on the adobe house? Why might a roof in Connecticut be steeper than a roof in the desert?

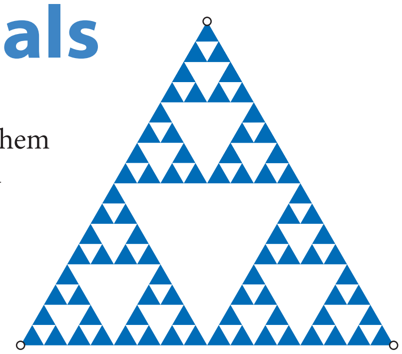

Adobe house, New Mexico


Pitched-roof house, Connecticut

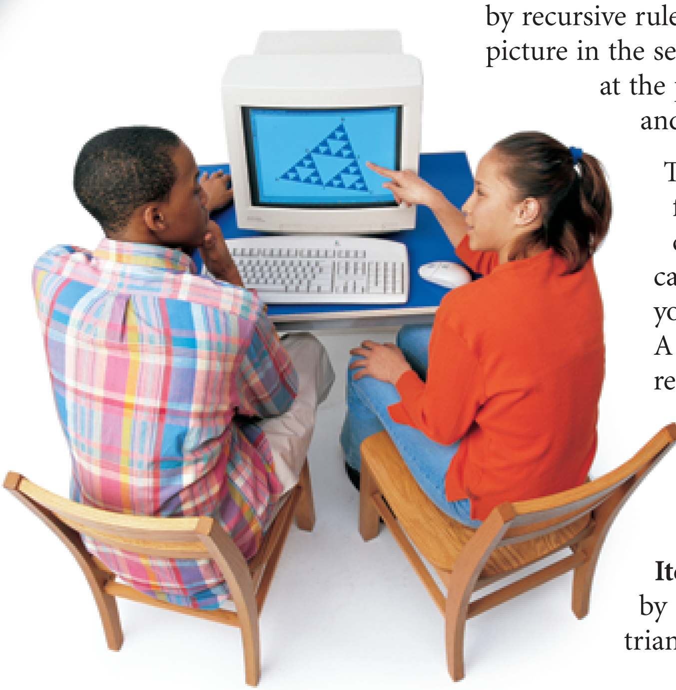
THE GEOMETER'S SKETCHPAD

Exploration

Patterns in Fractals



In Lesson 2.1, you discovered patterns and used them to continue number sequences. In most cases, you found each term by applying a rule to the term before it. Such rules are called **recursive rules.** Some picture patterns are also generated by recursive rules. You find the next picture in the sequence by looking at the picture before it and comparing that to the picture before it, and so on.



The Geometer's Sketchpad® can repeat a recursive rule on a figure using a command called **Iterate.** Using **Iterate,** you can create the initial stages of fascinating geometric figures called **fractals.** Fractals have **self-similarity,** meaning that if you zoom in on a part of the figure, it looks like the whole. A true fractal would need infinitely many applications of the recursive rule. In this exploration, you'll use **Iterate** to create the first few stages of a fractal called the **Sierpiński triangle.**

In this procedure you will construct a triangle, its interior, and midpoints on its sides. Then you will use **Iterate** to repeat the process on three outer triangles formed by connecting the midpoints and vertices of the original triangle.

This fern frond illustrates self-similarity. Notice how each curled leaf resembles the shape of the entire curled frond.

Activity

The Sierpiński Triangle

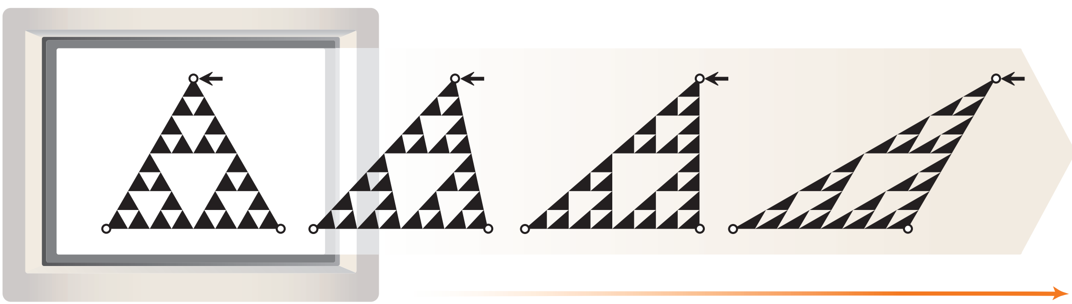
Procedure Note

Sierpiński Triangle Iteration

1. Open a new Sketchpad™ sketch.
2. Use the **Segment** tool to draw triangle *ABC*.
3. Select the vertices, and construct the triangle interior.
4. Select $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, in that order, and construct the midpoints *D, E,* and *F.*
5. Select the vertices again, and choose **Iterate** from the Transform menu. An Iterate dialog box will open. Select points *A, D,* and *F.* This maps triangle *ABC* onto the smaller triangle *ADF.*
6. In the Iterate dialog box, choose **Add A New Map** from the Structure pop-up menu. Map triangle *ABC* onto triangle *BED.*
7. Repeat Step 6 to map triangle *ABC* onto triangle *CFE.*
8. In the Iterate dialog box, choose **Final Iteration Only** from the Display pop-up menu.
9. In the Iterate dialog box, click Iterate to complete your construction.
10. Click in the center of triangle *ABC* and hide the interior to see your fractal at Stage 3.
11. Use Shift+plus or Shift+minus to increase or decrease the stage of your fractal.

Step 1 Follow the Procedure Note to create the Stage 3 Sierpiński triangle. The original triangle *ABC* is a Stage 0 Sierpiński triangle. Practice the last step of the Procedure Note to see how the fractal grows in successive stages. Write a sentence or two explaining what the Sierpiński triangle shows you about self-similarity.

Notice that the fractal's property of self-similarity does not change as you drag the vertices. For an interactive version of this sketch, visit www.keymath.com/DG .



Step 2 What happens to the number of shaded triangles in successive stages of the Sierpiński triangle? Decrease your construction to Stage 1 and investigate. How many triangles would be shaded in a Stage *n* Sierpiński triangle? Use your construction and look for patterns to complete this table.

Stage number	0	1	2	3	…	n	…	50
Number of triangles	1	3			…		…	

What stage is the Sierpiński triangle shown on page 135?

Step 3 Suppose you start with a Stage 0 triangle (just a plain old triangle) with an area of 1 unit. What would be the shaded area at Stage 1? What happens to the shaded area in successive stages of the Sierpiński triangle? Use your construction and look for patterns to complete this table.

Stage number	0	1	2	3	...	n	...	50
Shaded area	1	$\frac{3}{4}$			...		...	

What would happen to the shaded area if you could infinitely increase the stage number?

Step 4 Suppose you start with a Stage 0 triangle with a perimeter of 6 units. At Stage 1 the perimeter would be 9 units (the sum of the perimeters of the three triangles, each half the size of the original triangle). What happens to the perimeter in successive stages of the Sierpiński triangle? Complete this table.

Stage number	0	1	2	3	...	n	...	50
Perimeter	6	9			...		...	

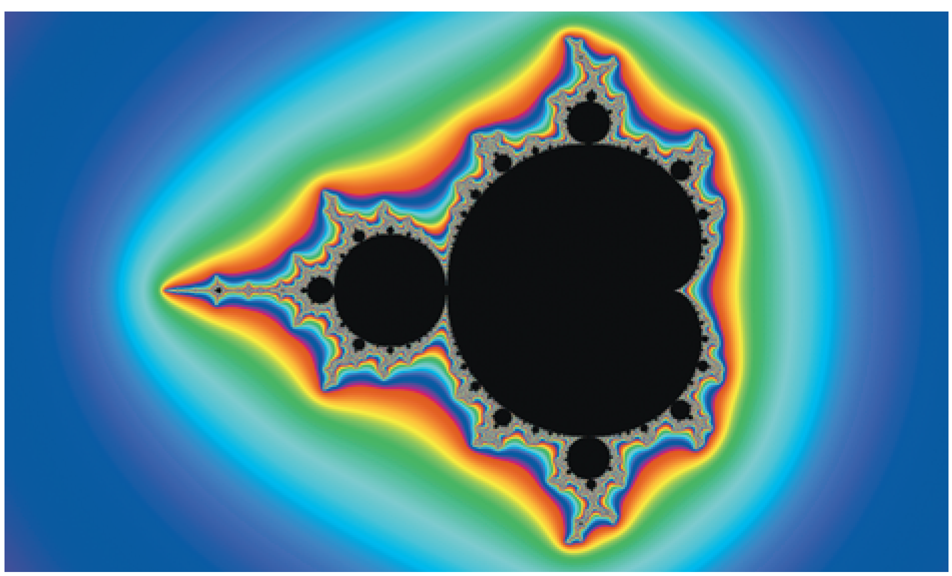
What would happen to the perimeter if you could infinitely increase the stage number?

Step 5 Increase your fractal to Stage 3 or 4. If you print three copies of your sketch, you can put the copies together to create a larger triangle one stage greater than your original. How many copies would you need to print in order to create a triangle two stages greater than your original? Print the copies you need and combine them into a poster or a bulletin board display.

Step 6 Sketchpad comes with a sample file of interesting fractals. Explore these fractals and see if you can use **Iterate** to create them yourself. You can save any of your fractal constructions by selecting the entire construction and then choosing **Create New Tool** from the Custom Tools menu. When you use your custom tool in the future, the fractal will be created without having to use **Iterate.**

The word *fractal* was coined by Benoit Mandelbrot (b 1924), a pioneering researcher in this new field of mathematics. He was the first to use high-speed computers to create the figure below, called the Mandelbrot set.

Only the black area is part of the set itself. The rainbow colors represent properties of points near the Mandelbrot set. To learn more about different kinds of fractals, visit www.keymath.com/DG .



CHAPTER 2 REVIEW

This chapter introduced you to inductive reasoning. You used inductive reasoning to observe patterns and make conjectures. You learned to disprove a conjecture with a counterexample and to explain why a conjecture is true with deductive reasoning. You learned how to predict number sequences with rules and how to use these rules to model application problems. Then you discovered special relationships about angle pairs and made your first geometry conjectures. Finally you explored the properties of corresponding, alternate interior, and alternate exterior angles formed by a transversal across parallel lines. As you review the chapter, be sure you understand all the important terms. Go back to the lesson to review any terms you're unsure of.

EXERCISES

1. "My dad is in the navy, and he says that food is great on submarines," said Diana. "My mom is a pilot," added Jill, "and she says that airline food is notoriously bad." "My mom is an astronaut trainee," said Julio, "and she says that astronauts' food is the worst imaginable." "You know," concluded Diana, "I bet no life exists beyond Earth! As you move farther and farther from the surface of Earth, food tastes worse and worse. At extreme altitudes, food must taste so bad that no creature could stand to eat. Therefore, no life exists out there." What do you think of Diana's reasoning? Is it inductive or deductive?

2. Think of a situation you observed outside of school in which inductive reasoning was used incorrectly. Write a paragraph or two describing what happened and explaining why you think it was poor inductive reasoning.

3. Think of a situation you observed outside of school in which deductive reasoning was used incorrectly. Write a paragraph or two describing what happened and explaining why you think it was poor deductive reasoning.

For Exercises 4–7, find the next two terms in the sequence.

4. 7, 21, 35, 49, 63, 77, ?, ?

5. Z, 1, Y, 2, X, 4, W, 8, ?, ? ⓗ

6. 7, 2, 5, -3, 8, -11, ?, ?

7. A, 4, D, 9, H, 16, M, 25, ?, ? ⓗ

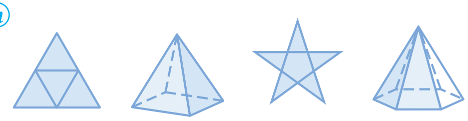
For Exercises 8 and 9, generate the first six terms in the sequence for each function rule.

8. $f(n) = n^2 + 1$

9. $f(n) = 2^{n-1}$ ⓗ

For Exercises 10 and 11, draw the next shape in the pattern.

10. 

11. ⓗ 

For Exercises 12–15, find the nth term and the 20th term in the sequence.

12.

n	1	2	3	4	5	6	…	n	…	20
$f(n)$	2	-1	-4	-7	-10	-13	…		…	

13.

n	1	2	3	4	5	6	…	n	…	20
$f(n)$	1	4	9	16	25	36	…		…	

14. ⓗ

n	1	2	3	4	5	6	…	n	…	20
$f(n)$	0	1	3	6	10	15	…		…	

15. ⓗ

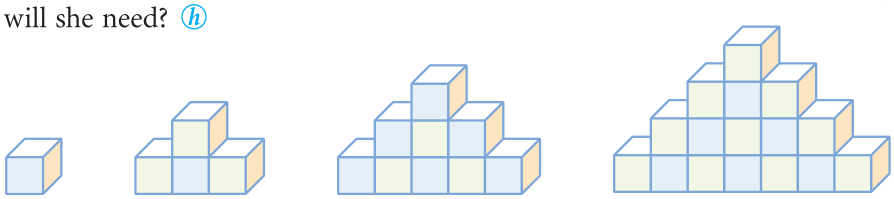
n	1	2	3	4	5	6	…	n	…	20
$f(n)$	1	3	6	10	15	21	…		…	

For Exercises 16 and 17, find a relationship. Then complete the conjecture.

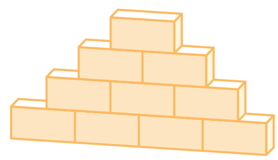
16. **Conjecture:** The sum of the first 30 positive odd whole numbers is _?_.

17. **Conjecture:** The sum of the first 30 positive even whole numbers is _?_. ⓗ

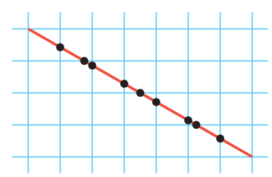
18. Viktoriya is a store window designer for Savant Toys. She plans to build a stack of blocks similar to the ones shown below but 30 blocks high. Make a conjecture for the value of the nth term and for the value of the 30th term. How many blocks will she need? ⓗ



19. The stack of bricks at right is four bricks high. Find the total number of bricks for a stack that is 100 bricks high.



20. For the 4-by-7 rectangular grid, the diagonal passes through 10 squares and 9 interior segments. In an 11-by-101 grid of squares, how many squares will the diagonal pass through? How many interior segments will it pass through? ⓗ



21. If at a party there are a total of 741 handshakes and each person shakes hands with everyone else at the party exactly once, how many people are at the party?

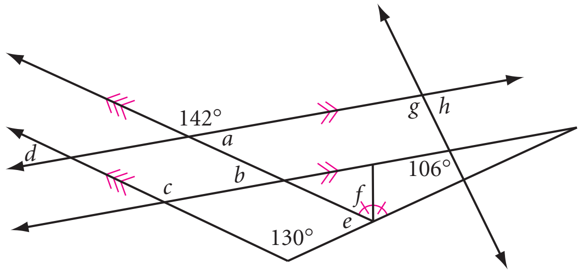
22. If 28 lines are drawn on a plane, what is the maximum number of points of intersection possible?

23. If a whole bunch of lines (no two parallel, no three concurrent) intersect in a plane 2926 times, how many lines are a whole bunch? ⓗ

24. If in a 54-sided polygon all possible diagonals are drawn from one vertex, they divide the interior of the polygon into how many regions?

25. How many sides does the polygon have if all possible diagonals drawn from one vertex divide the interior of the polygon into 54 regions?

26. Trace the diagram at right. Calculate each lettered angle measure.

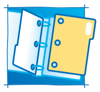


Assessing What You've Learned

WRITE IN YOUR JOURNAL

Many students find it useful to reflect on the mathematics they're learning by keeping a journal. Like a diary or a travel journal, a mathematics journal is a chance for you to reflect on what happens each day and your feelings about it. Unlike a diary, though, your mathematics journal isn't private—your teacher may ask to read it too, and may respond to you in writing. Reflecting on your learning experiences will help you assess your strengths and weaknesses, your preferences, and your learning style. Reading through your journal may help you see what obstacles you have overcome. Or it may help you realize when you need help.

- So far, you have written definitions, looked for patterns, and made conjectures. How does this way of doing mathematics compare to the way you have learned mathematics in the past?
- What are some of the most significant concepts or skills you've learned so far? Why are they significant to you?
- What are you looking forward to in your study of geometry? What are your goals for this class? What specific steps can you take to achieve your goals?
- What are you uncomfortable or concerned about? What are some things you or your teacher can do to help you overcome these obstacles?



KEEPING A NOTEBOOK You should now have four parts to your notebook: a section for homework and notes, an investigation section, a definition list, and now a conjecture list. Make sure these are up-to-date.

UPDATE YOUR PORTFOLIO Choose one or more pieces of your most significant work in this chapter to add to your portfolio. These could include an investigation, a project, or a complex homework exercise. Make sure your work is complete. Describe why you chose the piece and what you learned from it.